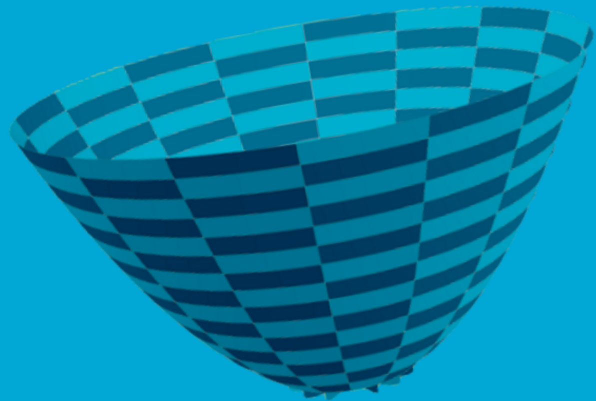
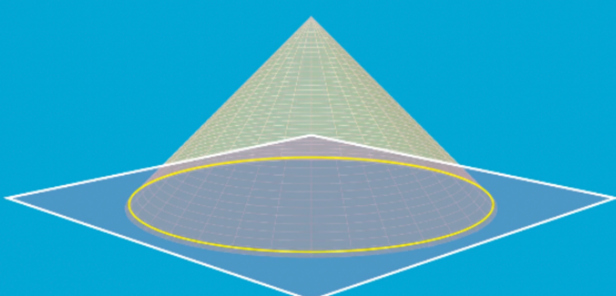
TMAS Academy

ACE AP Calculus AB 2024





$$\pi \int_a^b [f(x)^2 - g(x)^2]\,dx$$

- 150+ Problems
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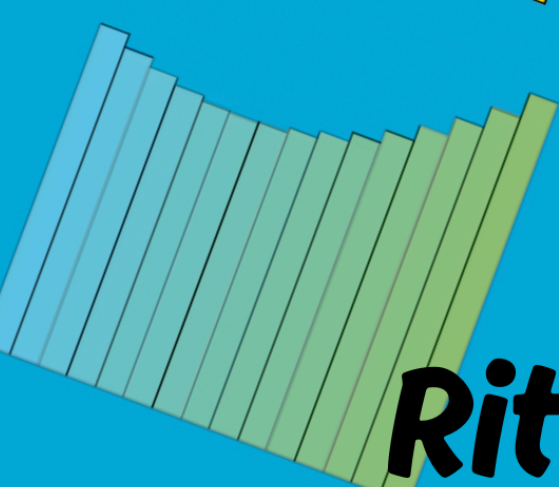

Ritvik Rustagi

Contents

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About TMAS Academy

This book is brought to you by me (Ritvik Rustagi). TMAS Academy, previously known as Explore Math, was started by me in 2020. TMAS stands for The Math and Science. Currently, I have written five free books for students around the world. Those books include the *ACE The AMC 10/12*, *ACE AP Physics 1*, *ACE AP Calculus AB*, *ACE AP Physics C: Mechanics*, and *ACE AP Calculus BC*. All of the books have been designed to make preparing for these exams efficient and accessible for everyone.
You can find more info about this program on my website linked below.
Website: https://www.tmasacademy.com/

Opportunities For You To Contribute To TMAS Academy

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https://forms.gle/VXGvj27UvcZPGhiJ8

Donations: If you want to assist me in my monthly payments to run this program which includes website costs, Overleaf costs (the platform used to write such books), and filming/editing costs, then please consider donating! For those that are willing to contribute, I have listed a few ways below. **Don't forget to write a message so I know who you are which will allow me to send you a thank you note**.

- You can donate through PayPal to the email: ritvikrustagi7@gmail.com
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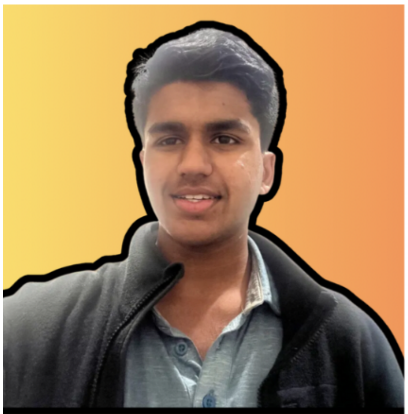
About The Author: Ritvik Rustagi

My name is Ritvik Rustagi, and I am a student at Prospect High School. Some information about me is that I enjoy doing math, physics, and programming.

Some of my qualifications include qualifying for USAJMO and USAMO (United States of America Mathematical Olympiad), qualifying for USAPHO (United States of America Physics Olympiad), achieving a gold medal in the finals round for MathCon in Chicago, and qualifying for the AIME several times.
During Covid, I discovered my passion for teaching math competition topics through my Youtube channel. It also allowed me to absorb these complicated topics more efficiently since teaching can help one improve their own skills. After that, I began my journey of writing various books for math competitions and AP courses. By October of 2023, I released my first major book for the AMC 10/12. In March 2024, I released several AP books which can all be found on the website.

This book has been written to help any student aiming to do well on the AP exam and the class itself. Due to the difficulty of this exam, a good guide is necessary with a rich problem set for students to practice with. This is what the book aims to do. Many students these days struggle to prepare for AP exams due to the vast amount of content. However, productive preparation can solve that problem. That is how I got 6 5s on the following AP exams in my sophomore year of high school: AP Physics 1, AP Calculus BC, AP Physics C: EM, AP Physics C: Mech, AP World History, and AP Statistics. Anyone can do it if they believe in themselves and choose the right resources to prepare with. Tons of problems are contained within this book with well written solutions. This will allow even the most inexperienced students to have a productive session of preparation while comprehending the problems and theory.



Benefits of Taking AP Exams

Preparing for AP exams such as the AP Calculus AB/BC, AP Physics 1, and AP Physics C is a great way to expand your knowledge. These exams go a step further to deepen your knowledge of subjects that you might have previously encountered. On top of that, you will learn many concepts that will be used throughout your life. It's a great learning experience and can give you the opportunity to enrich your journey. It also improves your problem solving skills which can serve as a life skill in many situations.

What if there is an error in the book?

There are possibilities for minor errors such as typos or a mistake in latex for some of the solutions to the problems. If that's the case, then please click on this link (`https://forms.gle/3mxZb4izUuBZLkmz5`) to report the mistake.

If you have any other questions or concerns, then please feel free to reach out to **ritvikrustagi7@gmail.com**

Credits

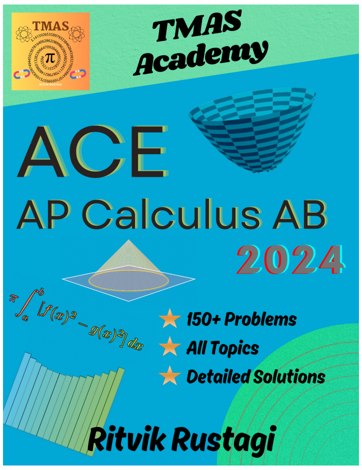
I would like to thank **The College Board** for their high quality problems that were used to teach concepts for this course.

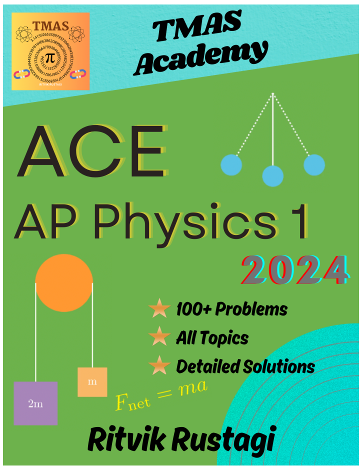
I would also like to thank **Evan Chen**, a PhD student at MIT (Massachusetts Institute of Technology), for his latex template which made it easy to format this book. I would also like to thank him for answering all of my questions regarding the format of this book.

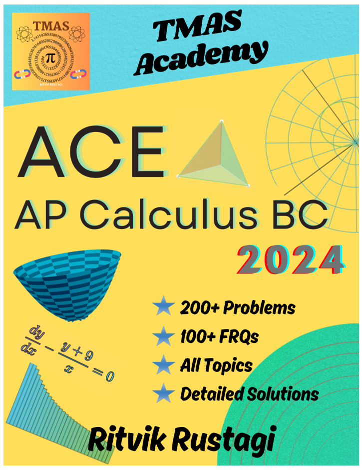
I would also like to thank my parents for supporting me with my goals and for everything else that they have done.

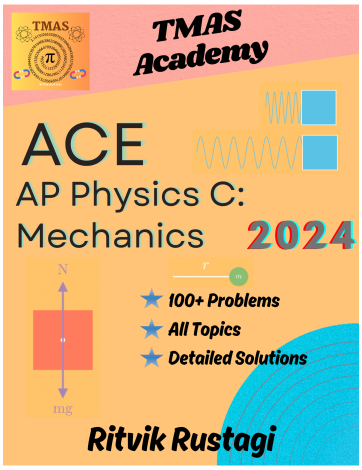
I would like to thank everyone else that has supported the content that I have made and encouraged me to continue to do so.

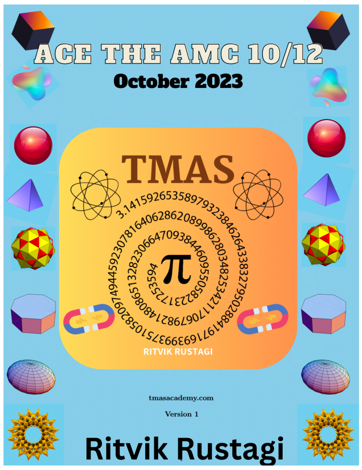
Other Important Resources











All five of these books were written by me. They can all be found on the TMAS Academy website. All five of the books are comprehensive and contain all the topics that you need to know. They have been designed to make your preparation productive through the vast number of official free response questions.

Make sure to check out the following playlists on the TMAS Academy youtube channel! These are important to learn all the topics that show up on the following AP exams: AP Physics 1, AP Calculus AB/BC, and AP Physics C: Mechanics.

AP Calculus AB/BC Playlist

AP Physics 1 Playlist

AP Physics C: Mechanics Playlist

Connect with the Author

Feel free to connect with me on Linkedin, Instagram, Discord, or through email!

I highly recommend joining the Discord server to access study and review sessions hosted in the server. You should also consider following the Instagram to access animations that will be posted there to allow you to learn.

Linkedin: https://www.linkedin.com/in/ritvik-638590210/

Instagram: https://www.instagram.com/ritvik_rustagi_tmas/

Discord: https://discord.gg/tmas-academy-1019082642794229870

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Unit 1 Limits and Continuity

§1.1 and 1.2 Defining Limits and Using Limit Notation

What does a limit do?
Many of you may know what an average rate of change is. An average rate of change defines the rate at which something changes over an interval (generally that interval is significantly wide). However, what if we want to find the rate of change for an infinitesimally small interval?

That is when limits come in. A limit allows us to define our instantaneous rate of change, which is what half of this course revolves around.

> **Note**
> **What is our notation for a limit?**
> $\lim_{x\to c} f(x) = R$
> This means that as x approaches c, $f(x)$ approaches R.
> You must remember the notation above and the way it's written. This will be seen throughout the entire AP exam and this course.

§1.3 and 1.4 Estimating Limit Values from Graphs and Tables

You must know what a one sided limit is.

> **Note**
> Our one-sided limit is the y-value that a function approaches as we approach a certain x value from either the left/right side.

Whenever we see a limit such as $\lim_{x\to c^-} f(x)$ (the left-hand limit), it is asking us for the value that $f(x)$ approaches as x approaches c from the NEGATIVE x axis side.

Similarly, when we see a limit such as $\lim_{x\to c^+} f(x)$ (the right-hand limit), then it is asking us for the value that $f(x)$ approaches as x approaches c from the POSITIVE x axis side.

Our two limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ can be different. They don't necessarily have to approach the same value.

Note

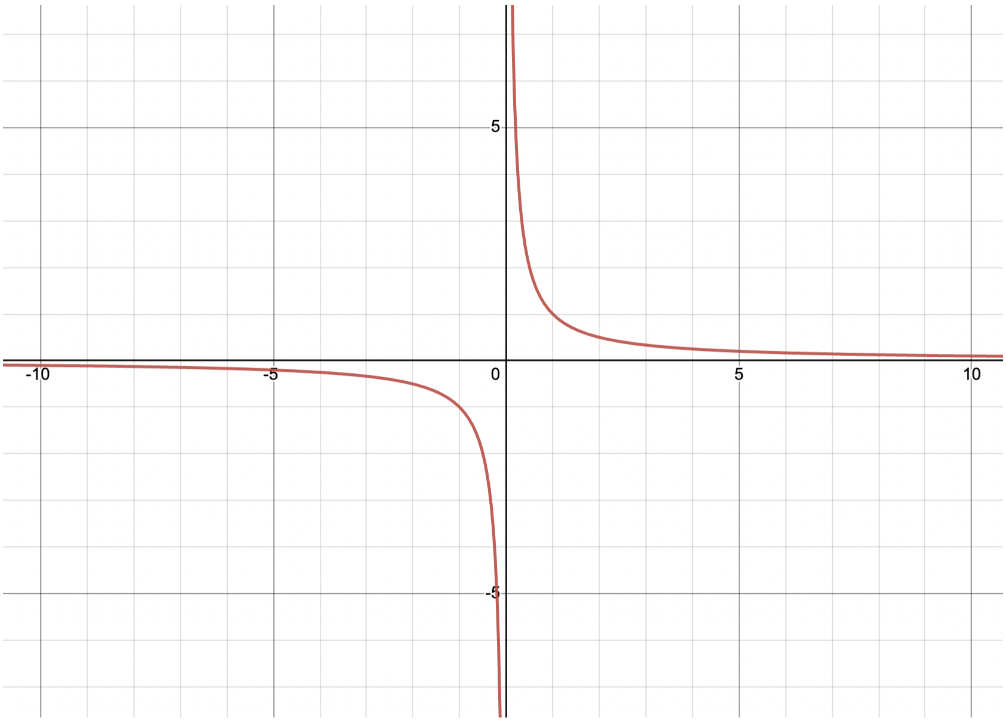
Although $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ can be different, $\lim_{x\to c} f(x)$ only exists if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$

This means that a limit only exists when the left-hand limit approaches the same value as the right-hand limit.

Problem — A function $f(x) = \frac{1}{x}$

Evaluate the following limits.
(a) $\lim_{x\to 0^-} f(x)$
(b) $\lim_{x\to 0^+} f(x)$
(c) $\lim_{x\to 0} f(x)$

Solution to part a: This part is asking us to find the value $f(x)$ approaches as x approaches 0 from the left. To be able to visualize this, we should first graph $\frac{1}{x}$.

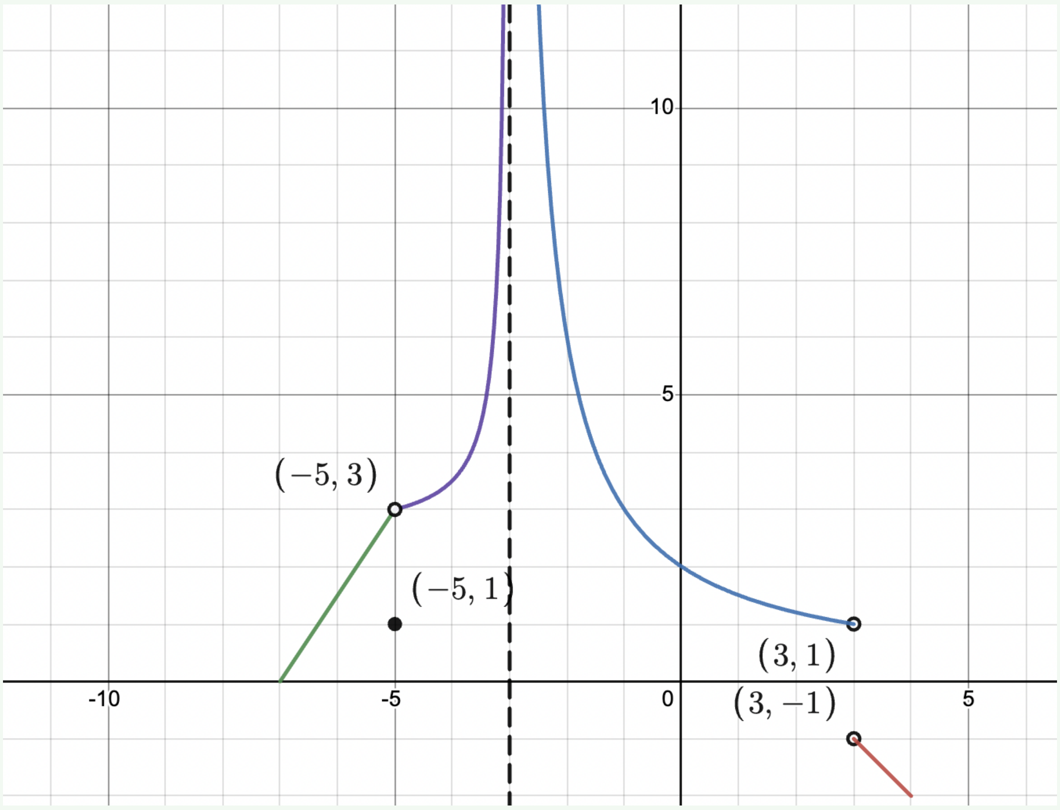


Clearly, $f(x)$ approaches $-\infty$ as x approaches 0 from the left-hand side.

Solution to part b: The right hand limit reaches $+\infty$ since the value of $f(x)$ increases drastically as x approaches 0 from the right side.

Solution to part c: Our left-hand limit is $-\infty$ while our right-hand limit is $+\infty$. Since the left-hand limit doesn't equal to our right-hand limit, our limit doesn't exist.

Problem —



Use the graph of $f(x)$ to find the values of the following:
(a) $f(-5)$
(b) $f(3)$
(c) $\lim_{x\to -3^-} f(x)$
(d) $\lim_{x\to -3^+} f(x)$
(e) $\lim_{x\to 3^-} f(x)$

Solution to part a: Some students will be confused regarding whether $f(-5)$ is 1 or 3. However, it is not 3 due to the open circle. The open circle means that that point isn't defined. However, a point with a black dot is defined. Thus, $f(-5) = 1$

Solution to part b: We will apply a similar concept as we did above. In this situation, both 1 and -1 at $x = 3$ are undefined due to the open circle. This means that $f(3)$ is undefined.

Solution to part c: We want to find what value $f(x)$ approaches as x approaches -3 from the left. This means x will approach -3 from the negative x axis side. Clearly, it approaches ∞ since the curve is pointing straight upwards along the vertical asymptote.

Solution to part d: We want to find what value $f(x)$ approaches as x approaches -3 from the right. This means x will approach -3 from the positive x axis side. Clearly, it approaches ∞ due to the vertical asymptote.

Solution to part e: We want to find what value $f(x)$ approaches as x approaches 3 from the left. Many students will be confused about whether the answer is 1 or -1. To efficiently solve this problem, we should be able to "trace" the curve that already exists with a pencil. We shouldn't lift our pencil up when finding the limit. This means we can trace our pencil along the curve that exists on the left of $x = 3$ making the limit approach 1 since it's a point on $f(x)$ when approached from the left.

Approaching limits through tables isn't a topic that shows up much and neither is it hard. You just have to observe the decimals and numbers to see what value the limit reaches. In almost all cases, it will be obvious. If you've watched the Unit 1 video on the TMAS Academy youtube channel, then you will know how to solve problems that involve a table.

§1.5, 1.6, and 1.7 Determining Limits Using Algebraic Properties

For limits that approach the same value for different functions, we can simply use simple mathematics for various properties of the limits.

Let's assume we have two limits: $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$

If k is a constant, then $\lim_{x\to c} k \cdot f(x) = k \cdot \lim_{x\to c} k \cdot f(x) = kA$

Similarly, $\lim_{x\to c} k \cdot g(x) = k \cdot \lim_{x\to c} k \cdot g(x) = kB$

There are a few more properties that you must know.

$$\lim_{x\to c} f(x) + g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = A + B$$

$$\lim_{x\to c} f(x) - g(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = A - B$$

$$\lim_{x\to c} f(x) \cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = A \cdot B$$

If $B \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{A}{B}$

All the properties above should remind you of simple arithmetic. Most basic math functions can apply to these limits as long as x approaches the same value.

Limits are often made tricky with piecewise functions. You can see examples in the Unit 1 rapid review video on the TMAS Academy Youtube Channel if you haven't already. You will see many examples after covering the theory for Unit 1. There will be plenty of free response questions to help you practice.

§1.8 Squeeze Theorem

I personally can't recall any instances of this on the AP exam. It's still useful to know just in case it shows up on the exam.

The squeeze theorem is often referred to as the "pinching" theorem.
If $g(x) \leq f(x) \leq h(x)$ and if $\lim_{x\to c} g(x) = A$ and $\lim_{x\to c} h(x) = A$, then $\lim_{x\to c} f(x) = A$

The first condition is that $f(x)$ must lie between the curves $g(x)$ and $h(x)$.

If $g(x)$ and $h(x)$ approach the same value, then $f(x)$ will also approach that same value as long as it lies between the two curves.

I would not recommend focusing too much on this topic (unless your teacher tests it). This topic rarely shows up on the AP exam. I can't recall any instance in which it did.

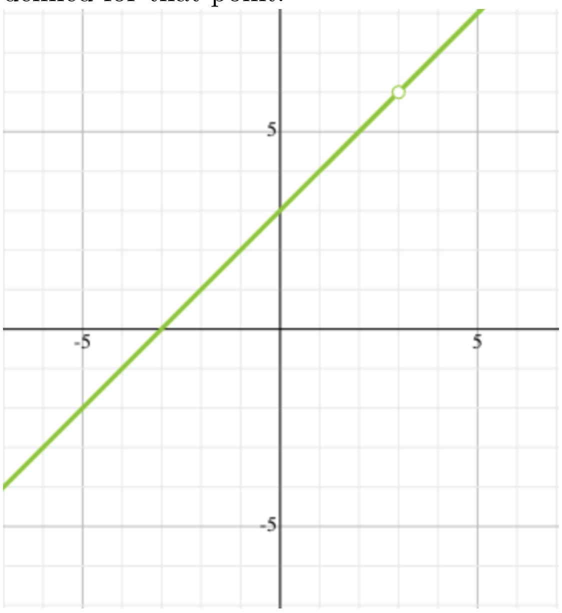
§1.9 Connecting Multiple Representations of Limits

There is no theory to learn in this section. It's just working with limits through different forms such as an equation, graph, or table. You need to know how to estimate a limit from a table, graph, and even an equation. You will see many examples in the FRQs that will be covered soon.

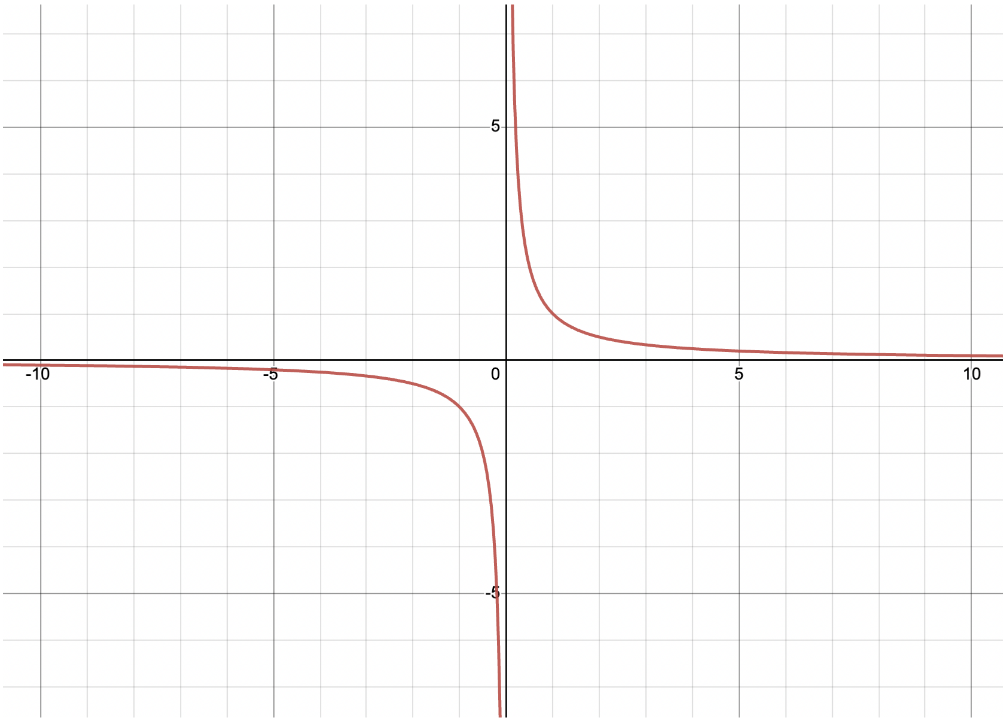
§1.10 Exploring Types of Discontinuities

There are 3 types of discontinuities. They is removable discontinuity, jump discontinuity, and infinite discontinuity.

A removable discontinuity is a point that is undefined and does not fit the rest of the graph.
An example of a removable discontinuity can be seen for the function $f(x) = \frac{x^2-9}{x-3}$
There is a removable discontinuity at $x = 3$, and it can be seen in the graph below. That is why there is an open circle at $x = 3$. That open circle means that the function isn't defined for that point.



An **infinite** discontinuity occurs due to an asymptote. $f(x)$ will approach either ∞ or $-\infty$ as x approaches some value towards that point of discontinuity.



In the example above, we have an infinite discontinuity as x approaches 0. The reason is that $f(x)$ approaches ∞ from the right side and $-\infty$ from the left side.

Note

How to find infinite discontinuity?
You will often be given a rational function. First, you must factor both the numerator and denominator. Then, you cancel out expressions that are the same at the top and bottom. The expressions you cancel out will give you your removable discontinuity. After cancelling, write out the denominator separately. Set the denominator equal to 0 and solve for the values that will satisfy the equation. Those values will give points of infinite discontinuity.

An example will now be shown.

Pretend we have a rational function $f(x) = \frac{x^2-3x-18}{x^2+8x+15}$. We want to find the points of discontinuity.
First, we factor the numerator and denominator to get $f(x) = \frac{(x+3)(x-6)}{(x+3)(x+5)}$
Since $x + 3$ cancels out, that will be our removable discontinuity due to a hole that occurs at that point. We have a removable discontinuity at $x = -3$.

After cancelling $x + 3$, we are left with $f(x) = \frac{x-6}{x+5}$
Now, we set the denominator to 0 to get $x + 5 = 0$. Clearly, $x = -5$, and that is our point of infinite discontinuity.

A jump discontinuity occurs when the graph simply "jumps" from one point to another with no connection in between. This also means that the left hand limit at that point doesn't equal to the right hand limit.
The image below will help summarize all 3 types of discontinuity.

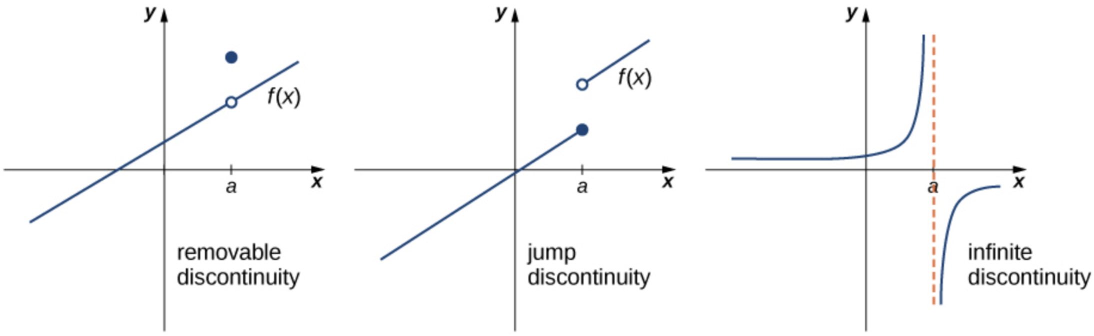


Image Credits: Full Credit goes to Math.LibreTexts for the image above

§1.11 Defining Continuity at a Point

To master this section, you just need to know one important rule.
For a function $f(x)$ to be continuous at point c, $f(c)$ must be defined and $\lim_{x\to c} f(x) = f(c)$

This topic often shows up on around 1 multiple choice question. It will often involve a piecewise function.

Problem —

$$f(x) = \begin{cases} x^2 + 3x + 4 & \text{if } x < -2 \\ x + 4 & \text{if } -2 \le x < 3 \end{cases} \tag{1.1}$$

Determine if $f(x)$ is continuous at $x = -2$.

Solution: We know that a point is continuous at $x = c$ when $\lim_{x\to c} f(x) = f(c)$.
This means that we want $\lim_{x\to -2} f(x) = f(-2)$

We will first evaluate the left and right hand limits.

$$\lim_{x\to -2^-} f(x) = (-2)^2 + 3 \cdot -2 + 4 = 2$$

$$\lim_{x\to -2^+} f(x) = -2 + 4 = 2$$

Since the left hand limit and right hand limit approach the same value (2), we know that $\lim_{x\to -2} f(x) = 2$.

Now we need to find $f(-2)$. We can plug in -2 into $x + 4$ since -2 is defined for the function $f(x) = x + 4$.
Clearly, $f(-2) = -2 + 4 = 2$

Since $\lim_{x\to -2} f(x) = f(-2) = 2$, our function $f(x)$ is continuous at $x = -2$.

§1.12 Confirming Continuity over an Interval

A function is continuous for an interval if it is continuous for every single point in the interval.

For problems from this section, you will often be given a function and an interval. You will be asked to find if the function is continuous for that interval.

The way to solve such problems is to find the points of discontinuity using the methods we discussed. Remember the 3 different types of discontinuity. If any of the points of discontinuity lie in the interval, then our function won't be continuous for that interval. However, if the points of discontinuity lie **outside** the interval, then our function will indeed be continuous for that interval.

If a problem asks us to find the domain for which a function is continuous, then make sure to leave out the points of discontinuity.

§1.13 Removing Discontinuities

This section is very similar to our discussion regarding removable discontinuities.
Remember what a removable discontinuity is and how to find one.

It's time for you to learn a more formal definition of removable discontinuity.
A removable discontinuity exists at $x = c$ if $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq f(c)$

Even though a hole occurs at a removable discontinuity, it still has a y value.
The reason is that the point of removable discontinuity gets 'cancelled out' from our function.

Let's say we have a function $f(x) = \frac{x^2-3x-18}{x^2+8x+15}$

We can factor it to get $f(x) = \frac{(x+3)(x-6)}{(x+3)(x+5)}$
We know that $x + 3$ cancels out. Thus, a removable discontinuity occurs at $x = -3$.

Our hole (point of removable discontinuity) still has a y-value. We can find it by cancelling out $x + 3$ to get $f(x) = \frac{x-6}{x+5}$.
Now, we can plug in $x = -3$ to get $f(-3) = \frac{-3-6}{-3+5} = -4.5$

Formally, $f(x)$ isn't defined at $x = -3$ due to a removable discontinuity. However, the limit still exists and can be found by cancelling out the common term which in this case was $x + 3$. Then, we can simply plug in our point of removable discontinuity to get -4.5 (which is the value $f(x)$ approaches as x approaches -3).

In summary, a limit will exist for points of removable discontinuity. However, the limit at that point won't be equal to the function's value. In addition, you can find the limit at ap oint of removable discontinuity by cancelling out like terms from the numerator and denominator before plugging in the desired point.

§1.14 Connecting Infinite Limits and Vertical Asymptotes

We already know that infinite discontinuities occur at vertical asymptotes. It causes $f(x)$ to approach either ∞ or $-\infty$ from either the left or right.

We can find vertical asymptotes by equating the denominator to 0. Before doing this, make sure to cancel out like terms from the numerator and denominator. After that you can set the denominator to 0 to find the vertical asymptotes.

A left-hand limit and right-hand limit towards a vertical asymptote will always be either $-\infty$ or ∞.

A double-sided limit on the other hand can either approach $-\infty$, ∞, or possibly not exist. The reason is that if the left-hand and right-hand limit approach two different values (such as $-\infty$ and ∞), then the two sided limit won't exist since we want the left and right hand limits to approach the same value for the double-sided limit to be defined.

§1.15 Connecting Limits at Infinity and Horizontal Asymptotes

By taking a limit to infinity for a function, we can find out its end behavior. This leads us to our topic of horizontal asymptotes.

The topic of horizontal asymptotes is often used with rational functions.

Note

If the denominator has a higher degree than the numerator, then $f(x)$ approaches 0 as x approaches ∞. That means a horizontal asymptote exists at $y = 0$.

If the denominator and numerator have the same degree, then $f(x)$ approaches the coefficient of the highest degree term in the numerator divided by the coefficient of the highest degree term in the denominator.
For example, if the highest degree term in the numerator is $3x^3$ while for the denominator it is $8x^3$, then the horizontal asymptote is $y = \frac{3}{8}$.

If the denominator has a lower degree than the numerator, then that means the numerator grows at a much faster pace than the denominator, then there is no specific value for the horizontal asymptote. Thus, none exists.

In some problems, there will be 2 horizontal asymptotes. This can occur when $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ approach two different values. However, if both approach the same value , then we only have one horizontal asymptote.

§1.16 Working with the Intermediate Value Theorem

The intermediate value theorem allows us to make an important justification.
In my opinion, this is the most important part of unit 1.
For the intermediate value theorem to work, there are some conditions that must be true.

First, $f(x)$ must be a continuous function for a certain closed interval $[a, b]$. Now, if d is a number that exists between $f(a)$ and $f(b)$, then there must be at least one number c between a and b such that $f(c) = d$.

This means that $f(c)$ will be between $f(a)$ and $f(b)$.

Unit 1 Practice Problems

There won't be many problems for this unit due to it's low yield. However, for other units expect to encounter much more problems.

Problem — 1982 AP Calculus AB FRQ (Modified)
Given that f is the function f defined by

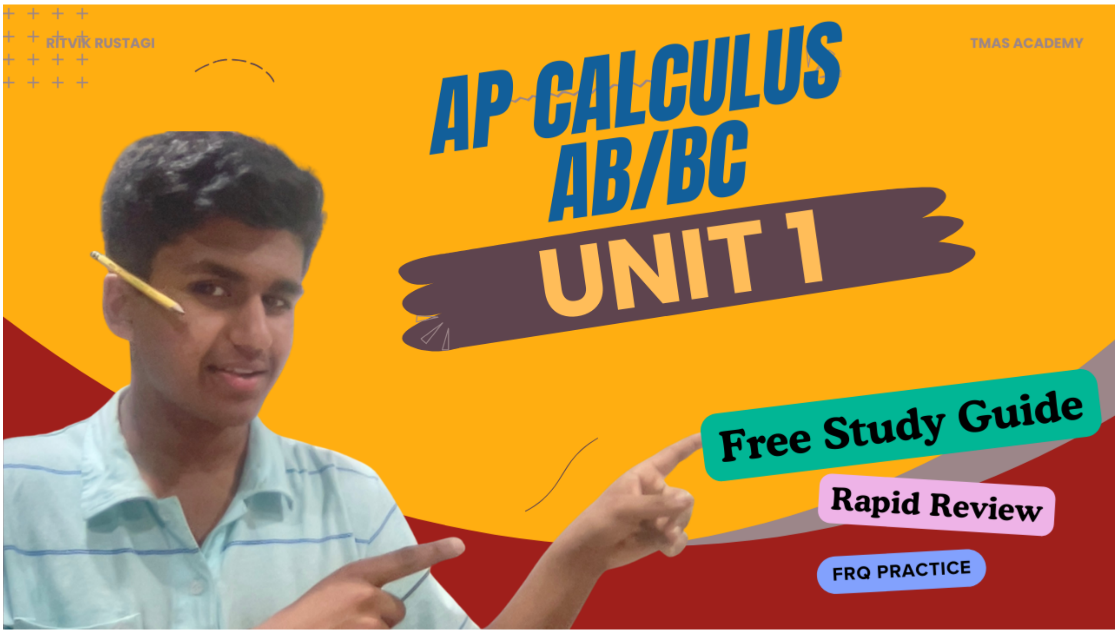
$$f(x) = \frac{x^3 - x}{x^3 - 4x}$$

(a) Find the $\lim_{x\to 0} f(x)$.

(b) Find the point at which a removable discontinuity occurs.

(c) Write an equation for each vertical and each horizontal asymptote to the graph of f.

Solution: Video Solution



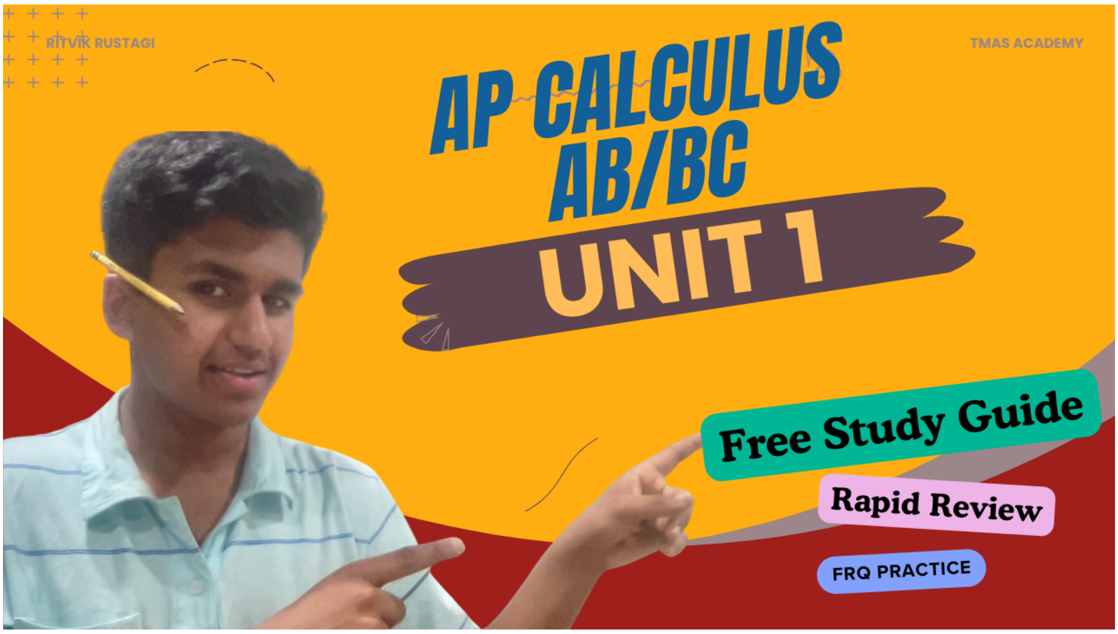
Problem — 1978 AP Calculus AB FRQ
Given the function f is defined by

$$f(x) = \frac{2x - 2}{x^2 + x - 2}$$

(a) For what values of x is $f(x)$ discontinuous?

(b) At each point of discontinuity found in a, determine whether $f(x)$ has a limit and, if so, give the value of the limit.

Solution: Video Solution



Problem — 2011 AP Calculus AB FRQ
Let f be a function defined by

$$f(x) = \begin{cases} 1 - 2\sin x \text{ for } x \le 0 \\ e^{-4x} \text{ for } x > 0 \end{cases} \tag{1.2}$$

Show that f is continuous at $x = 0$.

Solution: We know that a function will be continuous at point $x = c$ if $\lim_{x\to c} f(x) = f(c)$ Thus, we will first find $\lim_{x\to 0} f(x)$.

We will find the left-hand limit and right-hand limit.
$\lim_{x\to 0^-} f(x) = 1 - 2\sin 0 = 1$
Similarly, $\lim_{x\to 0^+} f(x) = e^{-4\cdot 0} = e^0 = 1$

Since the left-hand and right-hand limit both approach 1, the two-sided limit also approaches 1. $\lim_{x\to 0} f(x) = 1$

Now, we find $f(0)$.
$f(0) = 1 - 2\sin(0) = 1$
Since $\lim_{x\to 0} f(x) = f(0) = 1$, our function f is indeed continuous at $x = 0$.

Problem — 2003 AP Calculus AB FRQ
Let f be a function defined by

$$f(x) = \begin{cases} \sqrt{x+1} \text{ for } 0 \le x \le 3 \\ 5 - x \text{ for } 3 < x \le 5 \end{cases} \tag{1.3}$$

Is f continuous at $x = 3$? Explain why or why not.

Solution: We know a function is continuous at point c if the limit at that point equals

to the actual value.
This means that $\lim_{x\to c} f(x) = f(c)$

We will first find the left-hand limit and right-hand limit.
$\lim_{x\to 3^-} f(x) = \sqrt{3+1} = 2$
$\lim_{x\to 3^+} f(x) = 5 - 3 = 2$
The left-hand and right-hand limits both approach 2. This means that $\lim_{x\to 3} f(x) = 2$

Now, we must find $f(3)$. We use $f(x) = \sqrt{x+1}$ since 3 is in the bounds for that function. Clearly, $f(3) = \sqrt{3+1} = 2$.

Since $\lim_{x\to 3} f(x) = f(3) = 2$, our function f is continuous at $x = 3$.

Problem — 1986 AP Calculus AB FRQ
Let f be a function defined as follows

$$f(x) = \begin{cases} |x-1| + 2 \text{ for } x < 1 \\ ax^2 + bx \text{ for } x \geq 1 \end{cases} \tag{1.4}$$

Note, that a and b are constants above.
(a) If $a = 2$ and $b = 3$, is f continuous for all x? Justify your answer.

(b) Describe all values of a and b for which f is a continuous function.

Solution to part a: Whenever $x < 1$ and $x > 1$, $f(x)$ is always defined and continuous. However, the point $x = 1$ might lead to some issues since that's the point which can "connect" both parts of the graph.

Note, in problems such as this one, you will always be checking the "breakpoints" or endpoints.

We must check if $\lim_{x\to 1} f(x) = f(1)$
We will first find the left-hand and right-hand limits.
$\lim_{x\to 1^-} f(x) = |1-1| + 2 = 2$
$\lim_{x\to 1^+} f(x) = 2(1)^2 + 3(1) = 5$

Our left-hand and right-hand limits aren't equal. This means that $\lim_{x\to 1} f(x)$ doesn't exist.
Thus, $f(x)$ is continuous at all points except $x = 1$ since $\lim_{x\to 1} f(x)$ doesn't exist.

Solution to part b: We already know that f is continuous at all points except $x = 1$. Thus, we must find values of a and b that will make f continuous at $x = 1$.

We first want the limit to exist. The left-hand limit should approach the same value as the right-hand limit.

$$\lim_{x\to 1^-} f(x) = |1-1| + 2 = 2$$

$$\lim_{x\to 1^+} f(x) = a(1)^2 + b(1) = a + b$$

For the limit $\lim_{x\to 1} f(x)$ to exist, the left and hand right limits must equate. Thus,

$a + b = 2$.
Thus, a possible value of a and b is $(1, 1)$

Problem — 1989 AP Calculus AB FRQ
Let f be the function given by

$$f(x) = \frac{x}{\sqrt{x^2 - 4}}$$

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote to the graph of f.

(c) Write an equation for each horizontal asymptote to the graph of f.

Solution to part a: Since we have a fraction, we don't want the denominator to be 0. On top of that, since we have a square root function in the denominator, we know that the expression in the square root must be greater than 0.

Thus, $x^2 - 4 > 0$
We can solve this to get $x > 2$ and $x < -2$ for our domain.

Solution to part b: We can find our vertical asymptotes by factoring the denominator. Doing that gives

$$f(x) = \frac{x}{\sqrt{(x-2)(x+2)}}$$

Now, we set the denominator equal to 0.
$\sqrt{(x-2)(x+2)} = 0$
We can solve this to get that the vertical asymptotes are $x = 2$ and $x = -2$.

Solution to part c: We can find the horizontal asymptotes by taking the limit to ∞ and $-\infty$.
We know that the degree of the denominator is just 1 since $\sqrt{x^2}$ is x (which has a degree of 1).

Both the numerator and the denominator have the same degree of 1. Thus, to find the horizontal asymptotes, we only need to consider the coefficient of the term with the highest degree for both the numerator and denominator. For this problem, the coefficient will be 1.

$$\text{This means that } \lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}} = \frac{1}{1} = 1$$

This tells us that a horizontal asymptote exists at $y = 1$.

$$\text{Similarly, } \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 4}} = \frac{-1}{1} = -1$$

This means another horizontal asymptote exists at $y = -1$.

Problem — 2007 AP Calculus AB FRQ

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above give values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

Solution to part a: We plug in $x = 3$ into $h(x)$.
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
We plug in $x = 1$ to get $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

-5 lies between $h(1)$ and $h(3)$ and h is continuous. Thus, we can use the Intermediate Value Theorem (since our conditions are satisfied), and we know that there exists a value r for $1 < r < 3$ such that $h(r) = -5$.

Unit 2 Differentiation: Definition and Fundamental Properties

§2.1 Average and Instantaneous Rates of Change at a Point

Note

The average rate of change is simply a secant line that models the change over a big interval. You simply use your Δy (change in y) and Δx (change in x) to find the average rate of change.

For a function $y = f(x)$, the average rate of change between $x = a$ and $x = b$ is $\frac{f(b)-f(a)}{b-a}$

Our average rate of change can work for any interval length. However, sometimes we want to find the rate of change for a tiny interval that is infinitesimally small. That is when instantaneous rate of change comes in.

Note

Definition of a Derivative

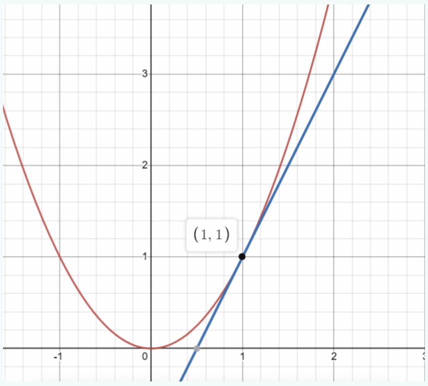
The derivative is one of the foundational tools for all of calculus and is based on the limit of a function as was discussed previously. We denote the derivative of a function in two ways. The derivative of y with respect to x can be expressed as $\frac{dy}{dx}$ or simply as y'.

There are two formal definitions of a derivative, both of which are below:

$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\frac{dy}{dx} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

The derivative can be thought of as a slope of the tangent line to any point on a curve. The image below shows the tangent line of $y = x^2$ at the point (1,1).



The picture above shows a tangent line to the graph of $y = x^2$ at $x = 1$

§2.2 Defining the Derivative of a Function and Using Derivative Notation

For a function $y = f(x)$, there are 3 main ways to represent the derivative.
They are $\frac{dy}{dx}$, y', and $f'(x)$.

The derivative is the same thing as a slope. The derivative of a function at a specific point is the slope of the graph at that point.

§2.3 Estimating Derivatives of a Function at a Point

You need to know how to use a calculator to find the derivative.

Some problems will give you a table and ask you to estimate the derivative using that. This just involves choosing the values closest to the point you want to find the derivative for. Then, you use those two values and find the slope between them. The value of that slope will be your estimate for the derivative.
If you don't understand what was said above, then don't worry. An example was shown in the AP Calculus AB/BC Rapid Review Unit 2 video on the TMAS Academy channel.

§2.4 Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

There are some cases where differentiation is not possible. We will explore them below:

1. At any point where a function is not continuous, differentiation is not possible. These locations of discontinuities include a removable discontinuity, an infinite discontinuity, and a jump discontinuity.

2. When there is a sharp turn in a function, for example, at $(1, 1)$ (as seen in the image below), differentiation would not be possible. The reason is that the left-hand and right-hand derivatives aren't equal.

3. If there is a vertical tangent to a point in a function, then the function is not differentiable at that point.

For the most part, you should be familiar with what removable, infinite, and jump discontinuities are. They were explained in the Unit 1 section in this book along with the AP Calculus AB/BC Rapid Review Unit 2 video on the TMAS Academy channel.

§2.5 Applying The Power Rule

We will explore some basic yet very powerful rules to take the derivative of a function. These rules will be used for differentiation throughout the course.

Note

If we have a function where the variable is raised to a power, for example $y = x^2$, then the power rule allows us to differentiate such an expression easily. We can bring the exponent down and make it the coefficient while reducing the exponent by 1. Thus, $\frac{d}{dx}(x^2) = 2x$.

The general form for the power rule is that $\frac{d}{dx}(x^n) = nx^{n-1}$.

Problem — Evaluate the following derivatives.
(a) $\frac{d}{dx}(x^5)$
(b) $\frac{d}{dx}(3x^3)$
(c) $\frac{d}{dy}(0.5y^3)$
(d) $\frac{d}{dx}(2x^{-4})$
(e) $\frac{d}{dx}(\frac{3}{x^2})$

Solution to part a: We apply the power rule to get $5x^4$

Solution to part b: We apply the power rule to bring the exponent of 3 down. This gives us $9x^2$.

Solution to part c: We bring the exponent of 3 down and multiply it to 0.5 to get $1.5y^2$

Solution to part d: Even though the exponent is negative, the power rule will still work in the same way. We bring -4 down and subtract 1 from the original exponent of -4. This gives us $-8x^{-5}$

Solution to part e: An important tip is to have all the terms with an exponent in the numerator. We can write $\frac{1}{x^2}$ as x^{-2}.
This means we want to find the derivative of $3x^{-2}$ which is just $-6x^{-3}$

§2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiples

Note

- The derivative of any constant is always 0. For example, $\frac{d}{dx}(4) = 0$.
- We can split derivatives into its respective addends. For example, $\frac{d}{dx}(3x^2 - 2x + 5) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(-2x) + \frac{d}{dx}(5)$. In general, this is represented as

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

- We can factor constants out of a derivative. For example, $\frac{d}{dx}(4x^2 + 2x) = 2 \cdot \frac{d}{dx}(2x^2 + x)$. The general form for this rule is

$$\frac{d}{dx}kf(x) = k \cdot \frac{d}{dx}f(x)$$

Now, you also need to know what a horizontal tangent line is. At a horizontal tangent line, the derivative/slope is 0.
That means for a function $f(x)$, you can write the equation $\frac{d}{dx}[f(x)] = 0$ to be able to find the points for which a horizontal tangent line exists.

Now, another concept that shows up a lot is the **normal line**. A normal line is perpendicular to the tangent line. For such problems, you will be given a specific point $x = c$ to work with.
Let's pretend we have a function $y = f(x)$. To be able to find the normal line, we must first find the derivative of $f(x)$.
The derivative at $x = c$ is $f'(c)$. This is also the slope of our tangent line.

We know that the slope of two lines that are perpendicular to each other will multiply to -1. This means that the normal line will have a slope of $-\frac{1}{f'(c)}$

Remember that the normal line goes through the point $c, f(c)$. This means that it intersects with the tangent line and the function $f(x)$ at that point. We can plug that point into an equation of a line $y = mx + b$. Doing so will allow us to find b (the y-intercept).

This may be confusing right now. However, don't worry since there will be examples later.

Remember, that differentiablity at a point means that it must also be continuous. Differentiability implies continuous behavior! However, this relation isn't true in the other direction. Something being continuous doesn't mean that it's also differentiable.

§2.7 and 2.10 Differentiating Important Functions

I combined topics 2.7 and 2.10 into this. These are some rules you should remember and memorize.
The following derivatives are of important functions and should be memorized as they are very common:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$

§2.8 and 2.9 Product Rule and Quotient Rule

Whenever we want to find the derivatives of the product or quotient of two functions, we must utilize the product rule and quotient rule.

- Product rule: If we have a function $F(x) = f(x)g(x)$, and we want to find $F'(x)$, we must use the product rule, which states:

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

For example, if we have $F(x) = x \sin x$, then

$$F'(x) = 1 \cdot \sin x + x \cdot \cos x = \sin x + x \cos x$$

- Quotient rule: If we have a function $F(x) = \frac{f(x)}{g(x)}$, then

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

For example, if we have $F(x) = \frac{x}{\sin x}$, then

$$F'(x) = \frac{1 \cdot \sin x - x \cdot \cos x}{\sin^2 x} = \frac{\sin x - x \cos x}{\sin^2 x}$$

Now before you move onto problems for unit 2, there are some more things you must know.

Note

Now sometimes a problem might ask you to find the equation of the tangent line. To solve this problem, you first find the slope of the tangent line using the derivative rules that you know.
Then, you write that slope in the equation $y = mx + b$.

Let's pretend that we have a function $f(x)$. The derivative of $f(x)$ which is $f'(x)$ is 2. Then, I can write my equation for the tangent line as $y = 2x + b$.

The tangent line will go through the point of tangency (the point we used originally). Thus, we can substitute that point into our equation for the tangent line to find our value of b (y-intercept) in $y = 2x + b$.
This might be confusing right now. Don't worry though since an example will be shown below to clarify this.

Problem — Equation of Tangent Line
What is the equation of the tangent line of $f(x) = 16 \cos x - 4$ at $x = \frac{\pi}{2}$

Solution: We will first take the derivative of this function to find the slope.
$f'(x) = -16 \sin x - 4$

Now I can plug in $x = \frac{\pi}{2}$ to find the exact slope.

$$f'(\frac{\pi}{2}) = -16 \sin(\frac{\pi}{2}) - 4 = -16 - 4 = -20$$

This means our slope is -20. We assume that the equation of our tangent line is $y = mx + b$
Since m represents the slope, we can plug -20 in to get $y = -20x + b$.

At $x = \frac{\pi}{2}$, the tangent line intersects with the function $f(x)$.

We can find that $f(\frac{\pi}{2}) = 16 \cos(\frac{\pi}{2}) - 4$. This simplifies to -4.

This means that our tangent line intersects $f(x)$ at the point $(\frac{\pi}{2}, -4)$

We can substitute this point into our tangent line equation: $y = -20x + b$.

That gives us $-4 = -20 \cdot \frac{\pi}{2} + b$. b can be found to be $10\pi - 4$

We can plug this into $y = -20x + b$ to find that the equation of our tangent line is $y = -20x + 10\pi - 4$

Sometimes, you will be given a piecewise function and asked to find if the function is differentiable at some specific point. The tricky part is that the function will be different depending on the bounds. However, just remember that for a derivative to exist, the **left-hand derivative** and the **right-hand derivative** must be equal.

This means that we can find both of those derivatives and see if they equate. If they do, then the function is differentiable for that certain point.

Unit 2 Practice Problems

Problem — 1977 AP Calculus BC FRQ
Let f and g and their inverses f^{-1} and g^{-1} be differentiable functions and let the values of f, g, and the derivatives f' and g' at $x = 1$ and $x = 2$ be given by the table below:

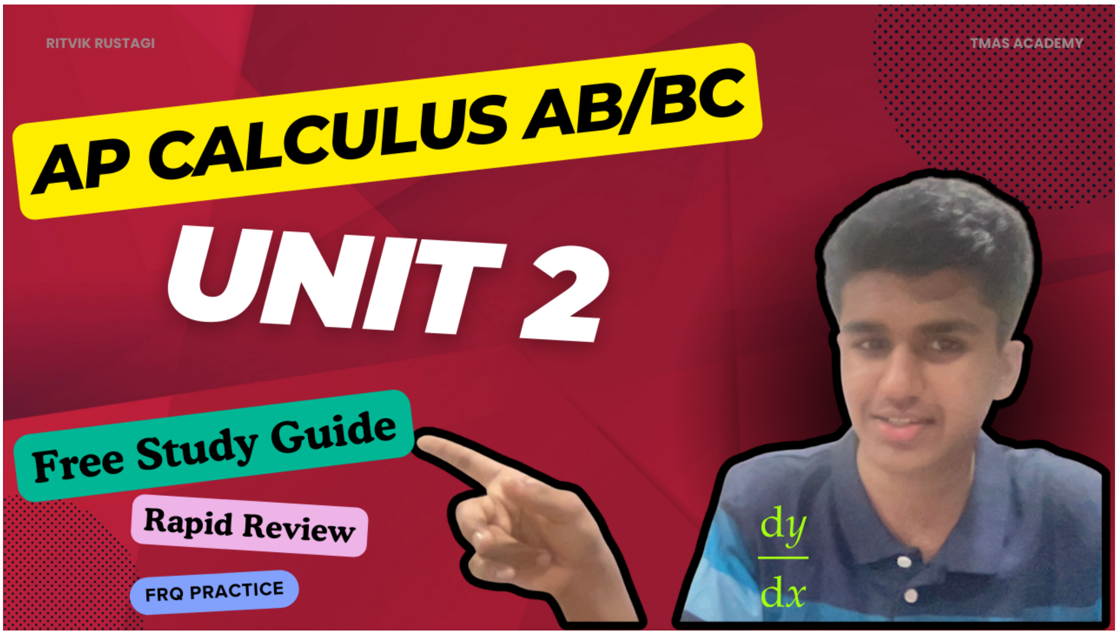
x	1	2
$f(x)$	2	3
$g(x)$	2	π
$f'(x)$	5	6
$g'(x)$	4	7

Determine the value of each of the following:
a. The derivative of $f + g$ at $x = 2$.

b. The derivative of fg at $x = 2$.

c. The derivative of $\frac{f}{g}$ at $x = 2$.

Solution: Video Solution



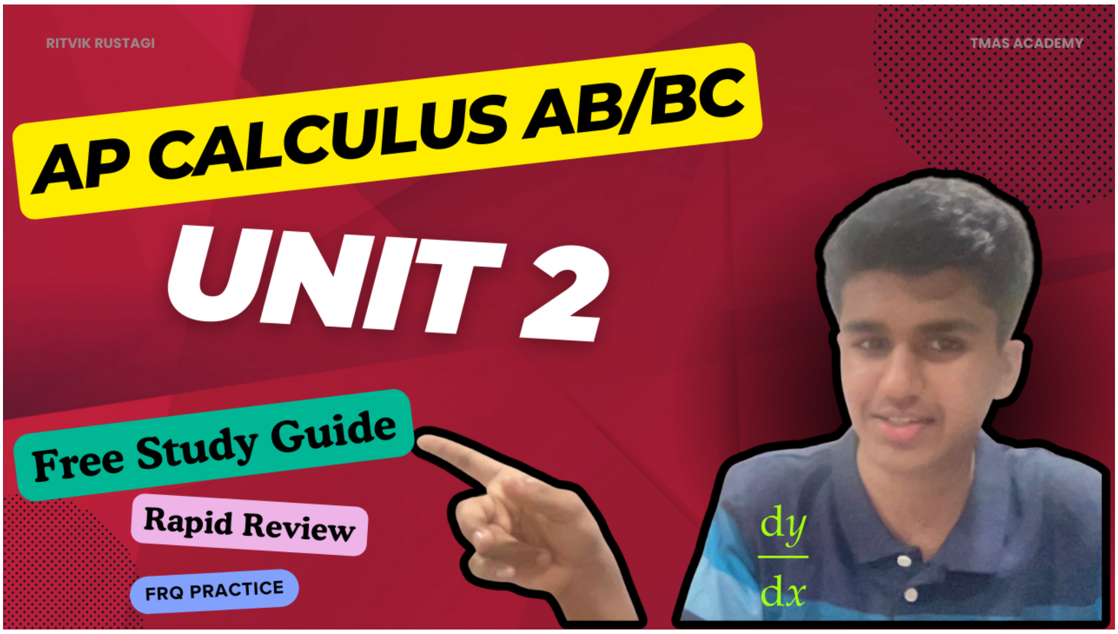
Problem — 1985 AP Calculus AB FRQ
Let f be the function given by $f(x) = \frac{2x-5}{x^2-4}$.
a. Find the domain of f.

b. Write an equation for each vertical and each horizontal asymptote for the graph of f.

c. Find $f'(x)$.

d. Write an equation for the line tangent to the graph of f at the point $(0, f(0))$.

Solution: Video Solution



Problem — 2003 AP Calculus AB FRQ (Modified)
Let f be a function defined by

$$f(x) = \begin{cases} \sqrt{x+1} \text{ for } 0 \leq x \leq 3 \\ 5 - x \text{ for } 3 < x \leq 5 \end{cases} \tag{2.1}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average rate of change of $f(x)$ on the closed interval $0 \leq x \leq 5$.

Solution to part a: We already covered this exact problem in Unit 1. Go back if you don't remember how to solve it.

Solution to part b: We know that the average rate of change of a function $f(x)$ for the closed interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$
In this problem, the average value is $\frac{f(5)-f(0)}{5}$
$f(5) = 5 - 5 = 0$
$f(0) = \sqrt{0+1} = 1$

We plug this in to find that the average rate of change is $\frac{0-1}{5} = -0.2$

Problem —
What is the average rate of change for $f(x) = x^3 + 4x^2 - 8$ for the interval $[2, 6]$.

Solution: We know that the average rate of change is simply $\frac{f(b)-f(a)}{b-a}$ for the interval between a and b.

The average rate of change for this problem is $\frac{f(6)-f(2)}{6-2}$ which is $\frac{f(6)-f(2)}{4}$.

Now we find the values of $f(6)$ and $f(2)$ so that we can plug it into the expression above.

$f(6) = 6^3 + 4 \cdot 6^2 - 8 = 352$
$f(2) = 2^3 + 4 \cdot 2^2 - 8 = 16$

Plugging this in gives that the average rate of change is $\frac{352-16}{4}$ which is **84**.

Problem —
Use the limit definition of derivative ($\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$) to find the derivative of $f(x) = 3x^2 + 7x + 8$

Solution: This problem can be done by directly plugging in $x+h$ into $f(x)$ to find $f(x+h)$.

$f(x+h) = 3(x+h)^2 + 7(x+h) + 8 = 3x^2 + 3h^2 + 6xh + 7x + 7h + 8$

Now we can plug this and $f(x) = 3x^2 + 7x + 8$ into $\frac{f(x+h)-f(x)}{h}$ to get

$$\frac{3x^2 + 3h^2 + 6xh + 7x + 7h + 8 - (3x^2 + 7x + 8)}{h}$$

This expression simplifies to $3h + 6x + 7$

This means our limit simplifies to $\lim_{h\to 0} 3h + 6x + 7$
We can directly plug in 0 for h to get that our derivative is

$$\boxed{6x + 7}$$

.
It is super important to know how to find the derivative using the definition of the derivative (limit method).

Problem —
Find the derivative of $f(x) = 16x^2 + 7\sqrt{x} - \frac{6}{x^2} + 7\pi^{100}$

Solution: We will find the derivative of each part separately, and then add up all the derivatives
The derivative of $16x^2$ is $32x$ because of the power rule.

Now to find the derivative of $7\sqrt{x}$, it's beneficial to convert this into exponential form since it makes things easier. $\sqrt{x}$ is simply $x^{\frac{1}{2}}$.
We can use the power rule on this to find that the derivative is

$$\frac{7x^{\frac{-1}{2}}}{2}$$

Now to find the derivative of $\frac{6}{x^2}$, it would be best to write it out as $6x^{-2}$.
The derivative of that is simply $-12x^{-3}$ using the power rule. $-12x^{-3}$ can also be written as $-\frac{12}{x^3}$

The derivative of $7\pi^{100}$ is 0 because π is a constant. It's not a variable.

Thus, the derivative of $f(x)$ is

$$32x + \frac{7x^{\frac{-1}{2}}}{2} - \frac{12}{x^3}$$

Problem — 2005 AP Calculus AB FRQ

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius, of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

Solution: This is an estimation problem. We will use the table to estimate the derivative. We will use the values of $T(x)$ at $x = 6$ and $x = 8$ since they are the closet points to the left and to the right of 7. Using those points will give a more accurate measure of the derivative when compared to other points from the table.

$T'(7) \approx \frac{T(8)-T(6)}{8-6} = \frac{55-62}{2} = -3.5\frac{\text{C}}{\text{cm}}$

Problem — 2018 AP Calculus AB FRQ
Let f be the function defined by $f(x) = e^x \cos x$.
(a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$.

Solution to part a: The average rate of change for a function $f(x)$ between points a and b is $\frac{f(b)-f(a)}{b-a}$ as already discussed.

We'll apply that formula to this problem. Since 0 and π are our endpoints for the interval in this case, our average rate of change is $\frac{f(\pi)-f(0)}{\pi}$

We just need to find the values of $f(\pi)$ and $f(0)$ to substitute in the expression above.

$f(\pi) = e^\pi \cos(\pi) = -e^\pi$
$f(0) = e^0 \cos(0) = 1$

Substituting these values in gives $\frac{-e^\pi-1}{\pi}$ which simplifies to $-\frac{1}{\pi}$.

Solution to part b: To find the slope of the tangent line, we simply take the derivative of $f(x)$ at point $x = \frac{3\pi}{2}$.

To find $f'(x)$, we need to use the product rule.
If we want the derivative of $h(x) \cdot g(x)$, then the derivative is $h'(x) \cdot g(x) + h(x) \cdot g'(x)$

We can apply that formula here for which $h(x) = e^x$ and $g(x) = \cos x$. We can use our derivative rules to quickly find that $h'(x) = e^x$ and $g'(x) = -\sin(x)$

We can plug this in to find that the derivative (slope of the tangent line at a point x) is $-e^x \sin(x) + e^x \cos(x)$.

Since that's the expression for $f'(x)$, we simply plug in $x = \frac{3\pi}{2}$ to find the slope at that point.

We get $-e^{\frac{3\pi}{2}} \sin(\frac{3\pi}{2}) + e^{\frac{3\pi}{2}} \cos(\frac{3\pi}{2})$. This simplifies to $e^{\frac{3\pi}{2}}$ which means the slope of our tangent line is $e^{\frac{3\pi}{2}}$.

Problem — 2019 AP Calculus AB FRQ
Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4+\frac{2}{3}(x-2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

Solution to part a: $h'(2)$ is the slope of the tangent line to the function $h(x)$ at the point $x = 2$. We are already given that the equation of the line tangent to $h(x)$ at the point $x = 2$ is $y = 4 + \frac{2}{3}(x - 2)$.

We can clearly see that the slope of the line $y = 4 + \frac{2}{3}(x - 2)$ is $\frac{2}{3}$. The reason is that for any line $y = mx + b$, the slope is simply m.
Our answer is thus $\frac{2}{3}$.

Solution to part b: To find the derivative of $a(x) = 3x^3h(x)$, we need to use the product rule.
The product rule states that the derivative of $f(x)g(x)$ is $f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

For this problem, we can say that $f(x) = 3x^3$ while $g(x)$ is $h(x)$.

$f'(x)$ is simply $9x^2$ from the power rule. Now we can start plugging in these values.
$a'(x) = 9x^2 \cdot h(x) + 3x^3 \cdot h'(x)$

Now since we want to find the derivative of $a(x)$ at point $x = 2$, we can substitute $x = 2$.

This simplifies our expression to $a'(2) = 36 \cdot h(2) + 24 \cdot h'(2)$

From part a, we already found that $h'(2) = \frac{2}{3}$. We are also given in the problem statement that $h(2) = 4$.
These substitutions give an answer of $36 \cdot 4 + 24 \cdot \frac{2}{3}$ which simplifies to **160**.

Problem — Quotient Rule
Find the derivative of the function $h(x) = \frac{x+8}{8-x}$.

Solution: We have to use the quotient rule in this problem.
The quotient rule states that the derivative of $\frac{f(x)}{g(x)}$ is

$$\frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

For this problem, we can say that $f(x) = x + 8$ and $g(x) = 8 - x$

This means that $f'(x)$ is 1 while $g'(x)$ is -1.

Making these substitutions gives $\frac{(8-x)\cdot 1-(x+8)\cdot -1}{(8-x)^2}$
This simplifies to $\frac{16}{(8-x)^2}$ which is the derivative.

Problem —
Is our function differentiable at the point $x = 3$.

$$f(x) = \begin{cases} 36x^2 - 3 & \text{if } x \geq 3 \\ 8x^3 + 5 & \text{if } x < 3 \end{cases}$$

Solution: We want the left and right-hand derivatives to be equal at $x = 3$.

The derivative of $f(x)$ whenever $x \geq 3$ is $72x$ using the power rule. This represents the right-hand derivative.
Similarly, the derivative of $f(x)$ whenever $x < 3$ is $24x^2$ using the power rule. This expression represents the left-hand derivative.

We can plug in our point $x = 3$ into both of the equations to get that both evaluate to 216.
Since both of them are equal, our function is differentiable at the point $x = 3$.

Problem — 1981 AP Calculus BC FRQ
Let f be a function defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 2 \\ \frac{x^2}{2} + k & \text{if } x > 2 \end{cases}$$

(a) For what values of k will f be continuous at $x = 2$? Justify your answer.

(b) Using the value of k found in part a, determine whether f is differentiable at $x = 2$. Use the definition of the derivative to justify your answer.

Solution to part a: The function f will be continuous at 2 if $\lim_{x\to 2} f(x) = f(2)$.
To find the value that the two-sided limit approaches, we must work with the left and right hand limits.

$$\text{Left-Hand Limit: } \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x + 1 = 2\cdot 2 + 1 = 5$$

$$\text{Right-Hand Limit: } \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2}{2} + k = \frac{2^2}{2} + k = 2 + k$$

For $\lim_{x\to 2} f(x)$ to exist, the left and right hand limits must approach the same value.
Thus, $k + 2$ must be equal to 5.
This means that $k = 3$.

On top of that, $f(2) = 2\cdot 2 + 1 = 5$.
At $k = 3$, the function f will be continuous at $x = 2$ because $\lim_{x\to 2} f(x) = f(2)$.

Solution to part b: f will be differentiable at $x = 2$ if the right-hand derivative is the same as the left-hand derivative.

$$\text{Left-Hand Derivative: } \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{2(x+h)+1-(2x+1)}{h}$$
$$= \lim_{h\to 0^-} \frac{2h}{h} = 2$$
$$\text{Right-Hand Derivative: } \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{\frac{(x+h)^2}{2}+k-(\frac{x^2}{2}+k)}{h}$$
$$= \lim_{h\to 0^+} \frac{h}{2} + x = x = 2$$

Also, for the expression above, after we find that the right hand derivative simply approaches x, we can plug in 2 since we are dealing with the point $x = 2$.

Thus, at the point $x = 2$, both the left and right hand derivatives approach 2. This means that the function is indeed differentiable at $x = 2$.

Unit 3 Differentiation: Composite, Implicit, and Inverse Functions

In this unit, we will be learning more crucial and special techniques for differentiation.

§3.1 Chain Rule

Note

You might be wondering what the chain rule is and the purpose. The chain rule is basically an important rule in derivatives when differentiating **composite** functions. Composite functions are in the form $f(g(x))$ where $f(x)$ and $g(x)$ are separate functions on their own.

$g(x)$ represents the inner function while $f(x)$ represents the outer function.

Chain Rule Formula

$$\frac{df(g(x))}{dx} = f'(g(x)) \cdot g'(x)$$

This means that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$.

This is extremely useful when you have an inner and an outer function. You can separate both which will allow you to find the derivative easily.

Another way the chain rule is written is $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ u represents the inner function. It's an intermediate variable used since it's a lot simpler to understand in comparison to a complicated inner function. If you want to use this notation for the chain rule, then make sure to watch the AP Calculus AB/BC Unit 3 Rapid Review video on the TMAS Academy youtube channel if you haven't already.

Problem —

What is the derivative of $f(x) = (x^3 + 8)^3$

Solution: In this problem, we will first identify our outer and inner function.
$x^3 + 8$ is our inner function and we can say that it is $g(x)$.
Our outer function is $h(x)$ which is x^3. This means $f(x) = h(g(x)) = (x^3 + 8)^3$

The derivative of $h(g(x))$ is simply $h'(g(x)) \cdot g'(x)$ using the Chain Rule.
To find $h'(g(x))$, we first find the derivative of $h(x)$. Then, we plug in $g(x)$ into that.
$h'(x) = 3x^2$
We plug in $g(x)$ into this to get that $h'(g(x)) = 3(x^3 + 8)^2$

We can find $g'(x)$ from the power rule to get that it is $3x^2$.

We multiply both of our expressions to find that the derivative of $f(x)$ is $9x^2(x^3+8)^2$

§3.2 Implicit Differentiation

The best way to master implicit differentiation is to practice a few problems.

Pretend a problem asks you to find the derivative of $y^2 = 4x^2 + 8x$.

You start off by differentiating with respect to x.
$\frac{d}{dx}[y = 4x^2 + 8x]$
Since you are differentiating with respect to x, you have to be careful with terms that contain y.
The derivative of y with respect to y is obviously 1. However, we must multiply this by $\frac{dy}{dx}$ to find it's derivative with respect to x. This can be proven using the chain rule.

For this specific scenario, the derivative of $4x^2+8x$ is simply $8x+8$ using the power rule.

Our equation can now be written as $\frac{dy}{dx} = 4x^2 + 8x$ which is the answer.
Implicit differentiation involves treating one variable as **the function** of the other. You might still be confused, so another example will be given to clarify it.

Problem — What is the derivative of $3y^5$ with respect to x.

Solution: We are supposed to find $\frac{d}{dx}[3y^5]$

We know from the chain rule that $\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$

We can apply that idea to this problem. We can say that $u = 3y^5$.
Since we want to find $\frac{du}{dx}$, we must simplify the right hand side. $\frac{du}{dy}$ is simply $15y^4$ due to the power rule. We leave $\frac{dy}{dx}$ the way it is.

This means that the derivative of $3y^5$ is simply $15y^4\frac{dy}{dx}$.

Note

Easier Method of Implicit Differentiation
If the variable I'm taking the derivative with respect to is different from the variable I'm differentiating, then I simply take the derivative of my variable normally. Then, I multiply it by the necessary derivative. This might be confusing, so I'll make it clear below.

Pretend I want to find $\frac{d}{dy}[x^3 + y^2 + 4x]$
I will use the power rule to differentiate each term normally, by ignoring the fact that my variables are different from the variable I'm differentiating with respect to. Doing so gives $3x^2 + 2y + 4$
Now I'll go back and multiply the terms that previously had x in them to $\frac{dx}{dy}$. This means that the derivative is $(3x^2 + 4) \cdot \frac{dx}{dy} + 2y$.

Overall, even though implicit differentiation is an application of the chain rule, you don't need to show the intermediate step that is taken using the variable u. You can differentiate normally and simply multiply some of the terms by $\frac{dy}{dx}$ or $\frac{dx}{dy}$ (depending on the scenario).

Often, implicit differentiation problems will give you an equation . You will first differentiate it and include $\frac{dy}{dx}$ or $\frac{dx}{dy}$ when necessary. Then, you will bring all terms with either $\frac{dy}{dx}$ or $\frac{dx}{dy}$ together to one side. This will allow you to solve for either $\frac{dy}{dx}$ or $\frac{dx}{dy}$ depending on the scenario.

This can be extremely confusing to understand, but trust me. You will understand it with the problems soon.

§3.3 Differentiating Inverse Functions

Differentiating inverse functions is easy as long as you know the important formula. If you don't, then it can be a struggle to derive it.

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

If we want to differentiate the inverse function at a certain value x, then you must know how to find the inverse of that x.

For example, if we know that $f(y) = x$, then that means $f^{-1}(x) = y$.
Thus, we can plug that into the derivative of the inverse function to get

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(y)}$$

Problem —
Find the derivative of $f^{-1}(a)$ when $f(x) = x^3 + 1$ at $a = 28$.

Solution: We already know that

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x)}$$

We can plug in 28 into this problem to simplify our formula above to
$\frac{d}{dx}[f^{-1}(28)] = \frac{1}{f'(f^{-1}(28)}$

To solve this problem, we have to evaluate $f'(f^{-1}(28)$ to simplify the fraction on the right.
$f^{-}1(28)$ can be found by equating 28 to $f(x)$. This gives us that $x^3 + 1 = 28$ which means $x = 3$.

Now we just need to find $f'(3)$. From the power rule, we know that $f'(x) = 3x^2$. $f'(3)$ must be 27.

This means our derivative is $\frac{1}{27}$ after plugging everything back in.

§3.4 Derivative of Inverse Trigonometric Functions

It's also important to know the derivative of your inverse trigonometric functions.
I would highly recommend memorizing them. Although it is possible to derive each formula, it can be time consuming at times, and I don't recommend it for the AP exam to maximize your chance of getting a 5.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

Often, differentiating inverse trigonometric functions will involve the chain rule. The reason is that you won't be given a simple problem such as $\frac{d}{dx}\sin^{-1}(x)$.
There will be an inner function such as x^2 that will make the problem $\frac{d}{dx}\sin^{-1}(x^2)$

Now, the outer function is $\sin^{-1}(x)$ while the inner function is x^2. We can apply the chain rule to find that the derivative is

$$\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

§3.5 Selecting Procedures for Calculating Derivatives

Concept wise, there is nothing in this section. Just remember ALL the derivative rules that you have learned so far. It will be used to a high extent throughout this unit and future units.

§3.6 Calculating Higher Order Derivatives

We have solved many problems that ask us to find $f'(x)$ or $\frac{dy}{dx}$. Both of those represent the **first derivative**.
Sometimes problems will ask you to find $f''(x)$ or $\frac{d^2y}{dx^2}$. Both of those represent the second derivative. You can find it by first finding the first derivative. Then, you differentiate the first derivative.

Remember that $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$ both represent the n-th derivative. You can continue to differentiate to find higher order derivatives.

Now we have learned our key derivative rules that will be used all throughout AP Calculus AB and BC. It's important to master these rules well if you want to be able to apply them to scenarios that will be shown in unit 4 and beyond. For now, it's time to practice some problems to drill these rules and to solidify your knowledge.

Unit 3 Practice Problems

Problem — 2014 AP Calculus AB FRQ
Grass clippings are placed in a bin, where they decompose. For $0 \leq t \leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \leq t \leq 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem

Solution to part a: The average rate of change for the interval $0 \leq t \leq 30$ is $\frac{A(30)-A(0)}{30-0}$

This is a FRQ from the calculator section. Thus, we just need to plug in our function of $A(t)$ into a calculator to evaluate $A(30)$
Using a calculator gives that $\frac{A(30)-A(0)}{30-0} = -0.197\frac{\text{lbs}}{\text{days}}$

Solution to part b: We can find $A'(15)$ by using our key calculator command for

derivatives.
Doing so gives that $A'(15) = -0.164$
Don't forget to interpret the meaning of the derivative in the context of the problem!
The amount of grass clippings in the bin decreases by $0.164\frac{\text{lbs}}{\text{days}}$ at the time $t = 15$ days.

Problem — 2000 AP Calculus AB FRQ
Consider the curve given by $xy^2 - x^3y = 6$
(a) Show that $\frac{dy}{dx} = \frac{3x^2y-y^2}{2xy-x^3}$

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

Solution to part a: This part is about implicit differentiation. We will differentiate the equation with respect to x.

$$\frac{d}{dx}[xy^2 - x^3y] = \frac{d}{dx}[6]$$

We must apply the product rule to each of the terms separately: xy^2 and x^3y

$$\frac{d}{dx}[xy^2] = \frac{d}{dx}[x] \cdot y^2 + x \cdot \frac{d}{dx}[y^2] = y^2 + x \cdot 2y\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$$

$$\frac{d}{dx}[x^3y] = \frac{d}{dx}[x^3] \cdot y + x^3 \cdot \frac{d}{dx}[y] = 3x^2y + x^3\frac{dy}{dx}$$

Also, $\frac{d}{dx}[6] = 0$ since the derivative of a constant is 0.

This means $y^2 + 2xy\frac{dy}{dx} - (3x^2y + x^3\frac{dy}{dx}) = 0$

Now, we can bring the terms that have $\frac{dy}{dx}$ in them to one side and the rest to the other.
Doing so gives $(2xy - x^3)\frac{dy}{dx} = 3x^2y - y^2$

We can divide both sides by $2xy - x^3$ to get

$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$$

Solution to part b: To find points that have an x-coordinate of 1, we plug in $x = 1$ into $xy^2 - x^3y = 6$.
Doing so gives $y^2 - y = 6$ which can be rearranged to get $y^2 - y - 6 = 0$
This is a polynomial that can be factored to $(y + 2)(y - 3) = 0$
This means that $y = -2$ or $y = 3$.

Our 2 possible points are $(1, -2)$ and $(1, 3)$

To find the equation of the tangent line, we will plug in each point into our expression for $\frac{dy}{dx}$. This will give us the slope. We also know that the point we test will lie on the

tangent line.
For $(1, -2)$ we get that

$$\frac{dy}{dx} = \frac{3(1)^2 \cdot -2 - (-2)^2}{2 \cdot 1 \cdot -2 - 1^3} = 2$$

This means that tangent line to the point $(1, -2)$ has a slope of 2.
Since a line is in the form $y = mx + b$, our equation so far is $y = 2x + b$

We know that the point $(1, -2)$ lies on this line. We can plug the point in to get $-2 = 2 + b$
Solving gives that $b = -4$. This means that our equation for the tangent line to the point $(1, -2)$ is $y = 2x - 4$

Now, we find $\frac{dy}{dx}$ for our other point which is $(1, 3)$.

$$\frac{dy}{dx} = \frac{3(1)^2 \cdot 3 - 3^2}{2 \cdot 1 \cdot 3 - 1^3} = 0$$

Using the equation of the form $y = mx + b$, we know our equation for the tangent line is $y = b$ so far (since the slope is 0).

We know that the point $(1, 3)$ must lie on this. We plug in 3 for y to find that $b = 3$. This means that our equation is simply $y = 3$.

Solution to part c: The tangent line will be vertical when the denominator is 0.
This means that $2xy - x^3 = 0$

We can factor the equation to get $x(2y - x^2) = 0$
Our solutions are $x = 0$ or $2y - x^2 = 0$

We will first test $x = 0$. We plug this point into the equation of the curve which is $xy^2 - x^3y = 6$
Doing so gives us $0 = 6$ which isn't a true statement. Thus, there is no solution for the $x = 0$ case.

Now, we will test $2y - x^2 = 0$ which is equivalent to $x^2 = 2y$
We can turn $x^2 = 2y$ to $y = \frac{x^2}{2}$

Now to test if there's a possible point of that form, we plug in $y = \frac{x^2}{2}$ into the equation of the curve which is $xy^2 - x^3y = 6$.

Doing so gives $x(\frac{x^2}{2})^2 - x^3 \cdot \frac{x^2}{2} = 0$

We can expand to get $\frac{x^5}{4} - \frac{x^5}{2} = 6$

We can simplify this to get $-\frac{x^5}{4} = 6$ which solves to $x^5 = -24$
We can solve this to get $x = \sqrt[5]{-24}$

Problem — 2012 AP Calculus AB FRQ
The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.

(a) Find $f'(x)$.

(b) Write an equation for the tangent line to the graph of f at $x = -3$.

(c) Let g be the continuous function defined by

$$g(x) = \begin{cases} f(x) \text{ for } -5 \leq x \leq -3 \\ 5 - x \text{ for } -3 < x \leq 5 \end{cases} \tag{3.1}$$

Is g continuous at $x = -3$. Use the definition of continuity to explain your answer.

Solution to part a: Our outer function is the square root function. The inner function is $25 - x^2$. This should immediately remind us of the chain rule.

$$f'(x) = -x(25 - x^2)^{-\frac{1}{2}}$$

Solution to part b: To find the equation of the tangent line, we will first find the slope of the tangent line to the graph of f at point $x = -3$

We already have an expression for $f'(x)$ which is $f'(x) = \frac{-x}{\sqrt{25-x^2}}$.

We plug in -3 to get $f'(-3) = \frac{3}{\sqrt{16}} = \frac{3}{4}$
This means that the slope of the tangent line is $\frac{3}{4}$

Thus, our equation for the tangent line is of the form $y = \frac{3}{4}x + b$
We know that the tangent line will intersect $f(x)$ at $x = 3$. This means we can the y-value for the point at $x = 3$.
$y = f(3) = \sqrt{25 - 3^2} = \sqrt{16} = 4$

This means that the tangent line goes through the point $(3, 4)$.
We plug that into the equation for the tangent line to get $4 = \frac{3}{4} \cdot -3 + b$
We can solve this for b to get $b = \frac{25}{4}$
This means that our equation for the tangent line is

$$y = \frac{3}{4}x + \frac{25}{4}$$

Solution to part c: A function will be continuous at a point $x = c$ if the limit equals to the actual value itself.
This means that $\lim_{x \to c} g(x) = g(c)$

We will first find our left-hand limit and right-hand limit.
$\lim_{x \to -3^-} g(x) = f(-3) = \sqrt{25 - (-3)^2} = \sqrt{16} = 4$
$\lim_{x \to -3^+} g(x) = -3 + 7 = 4$

Since the left-hand limit approaches the same value as the right-hand limit, we know that $\lim_{x\to 3} g(x) = 4$

Now, we will find $g(-3)$. Clearly, $g(-3) = f(-3) = \sqrt{25-(-3)^2} = \sqrt{16} = 4$

Since $\lim_{x\to -3} g(x) = g(-3) = 4$, our function $g(x)$ is indeed continuous at $x = -3$.

Problem — 2021 AP Calculus AB FRQ
Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y\sin x$ for $y > 0$.
(a) Show that $\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

Solution to part a: Whenever we are asked to find the derivative of an expression containing more than 1 variable, then we should think about implicit differentiation. We will evaluate $\frac{dy}{dx}$ and confirm if it's equivalent to $\frac{y\cos(x)}{4y-\sin(x)}$

We are supposed to find $\frac{d}{dx}[2y^2 - 6 = y\sin(x)]$
The derivative of $2y^2$ with respect to x is $4y \cdot \frac{dy}{dx}$ using our implicit differentiation techniques.

The derivative of -6 is obviously 0 since -6 is a constant.

The derivative of $y\sin(x)$ can be found using the product rule.
The product rule states that the derivative of $f(x)g(x)$ is $f'(x)\cdot g(x) + f(x)\cdot g'(x)$

In this problem, we can say that $f(x)$ is y while $g(x)$ is $\sin(x)$.

The product rule gives that the derivative is $\frac{dy}{dx}\cdot \sin(x) + y \cdot \cos(x)$

We can now plug this all back into our original expression which was $\frac{d}{dx}[2y^2 - 6 = y\sin(x)]$

We now get that it simplifies to $4y \cdot \frac{dy}{dx} = \frac{dy}{dx}\cdot \sin(x) + y\cdot\cos(x)$

We can bring like terms to one side to find that $\frac{dy}{dx}$ is $\frac{y\cos(x)}{4y-\sin(x)}$.

Solution to part b: $\frac{dy}{dx}$ represents our slope. We can plug in $(0, \sqrt{3})$ into our expression for $\frac{dy}{dx}$ to find the slope at that point.

We get $\frac{\sqrt{3}\cos(0)}{4\sqrt{3}-\sin(0)}$ which is $\frac{\sqrt{3}}{4\sqrt{3}}$ which simplifies to $\frac{1}{4}$.

Since our slope is $\frac{1}{4}$, we can plug this into the equation for the tangent line: $y = mx + b$.
The equation becomes $y = \frac{x}{4} + b$
We know that this equation intersects the point $(0, \sqrt{3})$ because the equation models the line tangent to $f(x)$.
Plugging in $(0, \sqrt{3})$ gives $\sqrt{3} = 0 + b$ which means b is $\sqrt{3}$.

Plugging this in gives that the equation is

$$y = \frac{x}{4} + \sqrt{3}$$

Solution to part c: Since we want to find when the tangent line is **horizontal**, we equate our expression for $\frac{dy}{dx}$ to 0. The reason is that a horizontal line will have a slope of 0.

$$\frac{dy}{dx} = \frac{y\cos(x)}{4y - \sin(x)} = 0$$

We know that the denominator can't be 0. Thus, $4y \neq \sin(x)$.

At the same time, $y\cos(x)$ must equal to 0 for the tangent line to be horizontal. However, this doesn't satisfy the bound of $y > 0$ which means we can disregard this case.

This condition means that either y must be 0 or $\cos(x)$ must be 0. For $\cos(x)$ to be 0, x has to be $\frac{\pi}{2}$ (this is the only value that fits the bounds and satisfies the conditions for x).

This means our value for x is $\frac{\pi}{2}$. We can plug this into the equation $\frac{y\cos(x)}{4y-\sin(x)} = 0$ to find the value of y that goes along with it.
Doing so gives that $y = 2$ when $x = \frac{\pi}{2}$.

This means that the answer is $(\frac{\pi}{2}, 2)$.

Problem — 2001 AP Calculus AB FRQ
The function f is differentiable for all real numbers. The point $(3, \frac{1}{4})$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6-2x)$

Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $(3, \frac{1}{4})$.

Solution: In this problem, we want to find the second derivative of our function. We are already given an expression for the first derivative. Thus, we can just differentiate the expression of the first derivative once.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}[y^2(6-2x)]$$

This is an implicit differentiation problem that also requires the product rule.

$$\frac{d}{dx}[y^2(6-2x)] = y^2\frac{d}{dx}[6-2x] + (6-2x)\cdot\frac{d}{dx}[y^2]$$
$$= (y^2 \cdot -2) + (6-2x)2y\cdot\frac{dy}{dx} = -2y^2 + 2y(6-2x)\frac{dy}{dx}$$

Now, we can plug in $\frac{dy}{dx} = y^2(6-2x)$ into our equation.
Doing so gives that the second derivative is $-2y^2 + 2y(6-2x)\cdot y^2(6-2x) = -2y^2 + 2y^3(6-2x)^2$

Now, we can plug in our point $(3, \frac{1}{4})$.
Doing so gives that

$$\frac{d^2y}{dx^2} = -2(\frac{1}{4})^2 + 2(\frac{1}{4})^3(6-6)^2 = -\frac{1}{8}$$

Problem — 2011 AP Calculus AB FRQ
Let f be a function defined by

$$f(x) = \begin{cases} 1 - 2\sin x \text{ for } x \leq 0 \\ e^{-4x} \text{ for } x > 0 \end{cases} \tag{3.2}$$

(a) Show that f is continuous at $x = 0$.

(b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

Solution to part a: The solution to part a from this exact problem was already covered in Unit 1. Go back to Unit 1 if you still can't solve this part. The entire solution is explained there.

Solution to part b: We will take the derivative of $1 - 2\sin x$ first. Doing so gives us $-2\cos x$.
Now we take the derivative of e^{-4x} to get $-4e^{-4x}$.
We can write this in the same format of the piece-wise functions with our bounds.

$$f(x) = \begin{cases} -2\cos x \text{ for } x < 0 \\ -4e^{-4x} \text{ for } x > 0 \end{cases} \tag{3.3}$$

Even though the original bound for the first equation was $x \leq 0$, we can exclude $x = 0$ because the problem says that $x \neq 0$.

Now, we will find the value of x that makes $f'(x) = -3$.
$-2\cos x$ can never be -3. The reason is that it will force $\cos x$ to be $\frac{3}{2}$ which is not possible since $\cos x$ must be between -1 and 1 inclusive.

Now we will test if $-4e^{-4x} = -3$
We can divide -4 from both sides to get $e^{-4x} = \frac{3}{4}$
We can take the natural log of both sides to get $-4x = \ln(\frac{3}{4})$

We can divide -4 from both sides to get $x = -\frac{1}{4}\ln(\frac{3}{4})$

Problem — 2006 AP Calculus AB FRQ
The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, f'(0) = -4, \text{ and } f''(0) = 3$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

Solution to part a: We must use all of the derivative rules that we have learned so far.

$$\text{Since } g(x) = e^{ax} + f(x), \text{ we get that } g'(x) = ae^{ax} + f'(x)$$

Now, we can plug in 0 to get $g'(0) = ae^{a \cdot 0} + f'(0) = a + f'(0)$

We are already given that $f'(0) = -4$, so we can plug that in to get $g'(0) = a - 4$

Since we know that $g'(x) = ae^{ax} + f'(x)$, we can differentiate this once to find the second derivative.

$$g''(x) = \frac{d}{dx}[ae^{ax} + f'(x)] = a^2e^{ax} + f''(x)$$

We can plug in 0 to get $g''(0) = a^2e^{a \cdot 0} + f''(0) = a^2 + f''(0)$

We are given that $f''(0) = 3$, so we can plug that in to get $g''(0) = a^2 + 3$

Solution to part b: We can apply the product rule to differentiate $h(x)$.

$$\begin{aligned} h'(x) = \frac{d}{dx}[h(x)] &= \cos(kx) \cdot \frac{d}{dx}[f(x)] + \frac{d}{dx}[\cos(kx)] \cdot f(x) \\ &= f'(x)\cos(kx) - k\sin(kx)f(x) \end{aligned}$$

We can now plug in 0 into the expression $h'(x) = f'(x)\cos(kx) - k\sin(kx)f(x)$

$$\begin{aligned} \text{Doing so gives } h'(0) &= f'(0)\cos(k \cdot 0) - k\sin(k \cdot 0)f(0) \\ &= f'(0)\cos(0) - kf(0)\sin(0) = f'(0) = -4 \end{aligned}$$

This means that the slope at $x = 0$ is -4. We can plug in -4 for m into $y = mx + b$
Doing so gives that the equation of the tangent line is $y = -4x + b$
At $x = 0$, $y = h(0) = \cos(0)f(0) = 2$

This means that the tangent line intersects the function h at $(0, 2)$.
We can plug in the point $(0, 2)$ into the equation for the tangent line which is $y = -4x + b$.

Doing so gives $2 = -4 \cdot 0 + b = b$

This means that $b = 2$

We can plug in $b = 2$ to get that the equation for the tangent line is $y = -4x + 2$

Problem — 2016 AP Calculus AB FRQ

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	–6	3	2	8
2	2	–2	–3	0
3	8	7	6	2
6	4	5	3	–1

The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph k at $x = 3$.

(b) Let $h(x) = \frac{g(x)}{f(x)}$. Find $h'(1)$.

Solution to part a: We are supposed to find the derivative of $k(x)$. We should be able to recognize that this problem involves the Chain Rule since $f(g(x))$ is a nested function.

The derivative of $k(x)$ or $f(g(x))$ is $f'(g(x)) \cdot g'(x)$. This means that $k'(x) = f'(g(x)) \cdot g'(x)$

We can plug in 3 to find the slope at $x = 3$.

$k'(3) = f'(g(3)) \cdot g'(3)$

We will first evaluate $f'(g(3))$. $g(3)$ is simply 6. Then, we can plug that in to simplify it to $f'(6)$. $f'(6)$ is simply 5.

Now we evaluate $g'(3)$ by using the table. $g'(3)$ is 2.

Thus, the slope of the line tangent to the graph of k at $x = 3$ is $5 \cdot 2$ which is 10.

We know that the equation of the tangent line can be written as $y = mx + b$
Our slope is 10, so the equation becomes $y = 10x + b$.

We know that the line tangent to the graph of k intersect at $x = 3$. This means that the y-coordinates for both $k(x)$ and the tangent line are the same at point $x = 3$.

Clearly, we are supposed to find $k(3)$ which is $f(g(3))$. We can use the values in the table to find that $f(g(3))$ is 4.

This means the point where the tangent line intersects the graph of k is $(3, 4)$.

We can plug in this solution pair into the equation of the tangent line: $y = 10x + b$

That gives us $4 = 30 + b$ which means b is -26.

Thus, the equation of the tangent line is $y = 10x - 26$

Solution to part b: We can apply the quotient rule to find $h'(1)$.

We first find $h'(x)$. Then, we'll plug in $x = 1$.
We can apply the quotient rule on $\frac{g(x)}{f(x)}$ to find that its derivative is $\frac{f(x)\cdot g'(x)-g(x)\cdot f'(x)}{(f(x))^2}$

$$h'(x) = \frac{f(x) \cdot g'(x) - g(x) \cdot f'(x)}{(f(x))^2}$$

We can plug in 1 into the equation above.

$$h'(1) = \frac{f(1) \cdot g'(1) - g(1) \cdot f'(1)}{(f(1))^2}$$

We use the values in our tables to find that $h'(1)$ is $\frac{-6\cdot 8-2\cdot 3}{(-6)^2}$ which simplifies to $-\frac{3}{2}$.

Problem — 2015 AP Calculus AB FRQ
Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

(a) Write an equation for the line tangent to the curve at point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

Solution to part a: To find the equation of the line tangent to the point $(-1, 1)$, we need to find the first derivative of the curve.

Lucky for us, it's already given in the problem that $\frac{dy}{dx} = \frac{y}{3y^2-x}$

We can plug in our point $(-1, 1)$ to find the derivative.

$$\frac{dy}{dx} = \frac{1}{3 \cdot 1^2 - (-1)} = \frac{1}{4}$$

Since we know that the equation of the tangent line can be modeled as $y = mx + b$, we can substitute our slope to get $y = \frac{x}{4} + b$.

We know that this tangent line intersects the curve at the point $(-1, 1)$. Thus, we can plug in that point into the equation to solve for b.

Doing so gives $1 = -\frac{1}{4} + b$ which means b is $\frac{5}{4}$.

Thus, the equation of our tangent line is $y = \frac{x}{4} + \frac{5}{4}$.

Solution to part b: The line tangent to the curve will be vertical when the slope is **undefined**. We know that the slope of the tangent line is represented as $\frac{y}{3y^2-x}$.

This will be undefined when the denominator is 0. This means that $3y^2 - x = 0$ which simplifies to $x = 3y^2$.

We can plug this into the equation of the curve which is $y^3 - xy = 2$. Substituting $x = 3y^2$ gives $y^3 - 3y^3 = 2$.
This can be simplifies to $y^3 = -1$ which means $y = -1$. We plug this into $x = 3y^2$ to find the x-coordinate which is 3.

This means that the tangent line is vertical at the point $(3, -1)$.

Solution to part c: $\frac{d^2y}{dx^2}$ represents the second derivative. Since we are already given the equation of the first derivative, we simply take the derivative of that once to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[\frac{y}{3y^2 - x}]$$

We apply the quotient rule to this. We also have to be mindful of our implicit differentiation technique. For example, the derivative of y^2 with respect to x is $2y \cdot \frac{dy}{dx}$.

Applying the quotient rule gives that the second derivative is

$$\frac{(3y^2 - x)\frac{dy}{dx} - y(6y\frac{dy}{dx} - 1)}{(3y^2 - x)^2}$$

We already know that $\frac{dy}{dx}$ is $\frac{y}{3y^2-x}$. We can plug in our coordinates $(-1, 1)$ to find that $\frac{dy}{dx}$ is $\frac{1}{4}$.

We can plug in our coordinate $(-1, 1)$ and $\frac{1}{4}$ for $\frac{dy}{dx}$ to find that the second derivative is $\frac{1}{32}$.

Problem — Inverse Trigonometric Derivatives
Find the derivative of $f(x) = \sin^{-1}(2x^2)$

Solution: We can apply the chain rule on this problem since our outer function is $sin^-1(x)$ while the inner function is $2x^2$.

We can say that $g(x)$ is $\sin^{-1}(x)$ while $h(x)$ is $2x^2$.
This means that $f(x) = g(h(x))$.

We can use the chain rule to find that $f'(x) = g'(h(x)) \cdot h'(x)$.

To find $f'(x)$, we first find $g'(h(x))$. To do this, we first find the $g'(x)$ and then plug in $h(x)$ into that.

$g'(x)$ is $\frac{1}{\sqrt{1-x^2}}$ from our inverse trigonometric derivative formulas.

Now we plug in $h(x) = 2x^2$ instead of x. This means that $g'(h(x))$ is $\frac{1}{\sqrt{1-(2x^2)^2}}$ which is $\frac{1}{\sqrt{1-4x^4}}$

The second part of our chain rule expression is simply $h'(x)$ which is $4x$ because of the power rule.

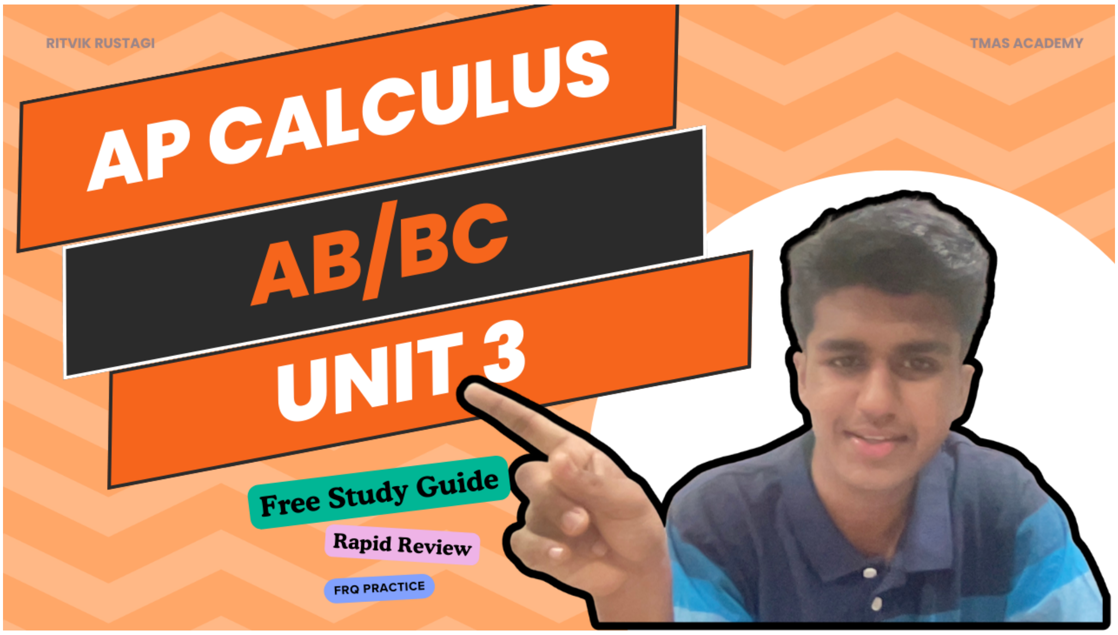
Multiplying both expressions gives our derivative which is $\frac{4x}{\sqrt{1-4x^4}}$

Problem — 2004 AP Calculus AB FRQ
Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$

(a) Show that $\frac{dy}{dx} = \frac{3y-2x}{8y-3x}$

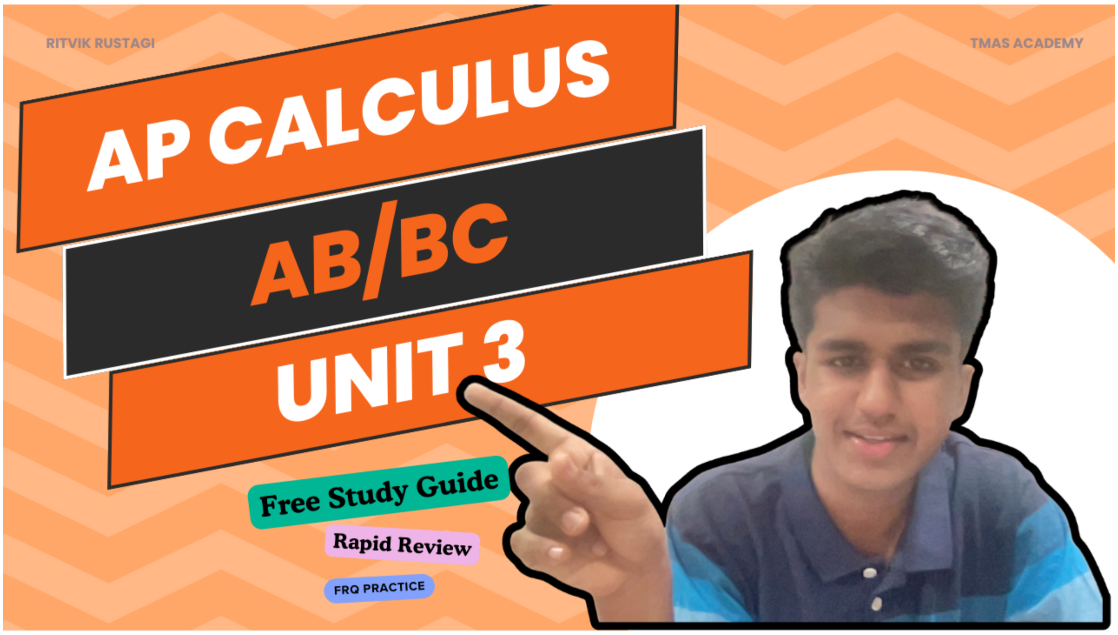
(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

Solution: Video Solution



Problem — 1990 AP Calculus AB FRQ
Let f be the function given by $f(x) = \ln \frac{x}{x-1}$.
Find the value of the derivative of f at $x = -1$.

Solution: Video Solution



Unit 4 Contextual Applications of Differentiation

§4.1 Interpreting the Meaning of the Derivative in Context

Most of the interpreting problems ask you to interpret the first or second derivative. To successfully interpret it, you just need to know what's happening in the problem for a certain, specific scenario.

Most of the interpreting problems are pretty repetitive and you can get away with memorizing a format for them.
Below there will be two templates listed that can be used.

> **Note**
> **Interpreting First Derivative**
> The instantaneous rate of change of [Context] in [Units] at [specific point x] is

You'll see some interpretation examples at the end of the unit through the plethora of free response questions.

> **Note**
> **Interpreting Second Derivative**
> The rate of [context] at time [t] is decreasing/increasing at $H''(x)$ [units]

For example, if $H''(t) = 4$ at time $t = 3$ (and $H(t)$ represents the height at time t), then we can interpret the second derivative by saying: the rate at which the rate of change of height $(H'(t))$ changes at time $t = 3$ is increasing at a rate of $H''(t) = 4$.

For interpreting problems, you should carefully read the problem to understand the given scenario. As long as you explain the first derivative and second derivative in the context of the problem, you should be able to earn the point for it.

§4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration

The position at time t is represented as $x(t)$. The velocity at time t is represented as $v(t)$. The acceleration at time t is represented as $a(t)$.

The first derivative of position gives you velocity. The first derivative of velocity gives you acceleration while the second derivative of position also gives you acceleration.

$$x'(t) = v(t)$$

$$x''(t) = a(t)$$

$$v'(t) = a(t)$$

The above three equations are something you must know. They represent the relationship between position, velocity, and acceleration.

Note

When is an object at rest? When is it moving to the left? When is it moving to the right?

An object is at rest when $v(t) = 0$. It is moving to the left when $v(t) < 0$. It is moving to the right when $v(t) > 0$.

Note

When is an object speeding up/slowing down?
The most commonly made error here is to think that it's speeding up when acceleration is positive and it's slowing down when acceleration is negative. However, this is **false**.

An object is speeding up when $v(t)$ and $a(t)$ have the SAME sign (either + or -)

On the other hand, an object is slowing down when $v(t)$ and $a(t)$ have different signs.

Problem — 2017 AP Calculus AB FRQ
Two particles move along the x-axis. For $0 \leq t \leq 8$, the position of particle P at time t is given by $x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \leq t \leq 8$, when is particle P moving to the left?

(b) Part b has been removed.

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

Solution to part a: The particle is moving to the left when the velocity is negative. Since we are given the equation of the position, we can take the derivative of that to find the equation of the velocity. To differentiate $x_P(t) = \ln(t^2 - 2t + 10)$, we need to use the chain rule.

$$v_p(t) = \frac{dx_p(t)}{dt} = \frac{2t-2}{t^2-2t+10}$$

Since we are supposed to find when the particle is moving to the left, we equate the equation for velocity to 0. Then, we solve for possible values of time that make the expression 0 and see for what range the velocity will be negative.

Clearly, the denominator will always be positive for the interval $0 \leq t \leq 8$.

This means we can just focus on the numerator. The expression will be 0 when $2t - 2 = 0$. This means our value of t is 1. We can see that when $t < 1$, our velocity will be negative.

This means our answer is $0 \leq t < 1$

Solution to part c: To find if the speed is increasing or decreasing, we need to know the signs of acceleration and velocity.

We can find acceleration by taking the derivative of $v_Q(t) = t^2 - 8t + 15$
$a_Q(t) = \frac{dv_Q(t)}{dt} = 2t - 8$

To find the acceleration and velocity at time 2, we can simply plug in 2 into $a_Q(t)$ and $v_Q(t)$. When we plug in 2 into $a_Q(t)$, we notice that the sign of the acceleration is negative. However, when we plug in 2 into the expression for $v_Q(t)$, the sign of velocity is positive.

Since acceleration and velocity have different signs, the speed is **decreasing** at time 2.

§4.3 Rates of Change in Applied Contexts Other Than Motion

In general, if a function is increasing at a certain point, then the sign of its derivative at that point will be positive.

For example, let's say we have a function $h(x)$ representing the height. If $h'(x)$ is positive, then the height is increasing. If $h'(x)$ is negative, then the height is decreasing.

If $h''(x)$ is positive, then the rate at which height changes increases.

Similarly, if $h''(x)$ is negative, then the rate at which height changes decreases.

In general, this can be used for other scenarios.

§4.4 Related Rates

Note

Related Rates Basics

Related rates is one of the most important topics that a lot of people struggle with. Related rates problems are common when you have many variables, but you can relate some of them. For example, area of a circle (A) and radius (r) are both different variables. However, you can rewrite the area variable (A) as πr^2

Related rates problems often use the topic of **chain rule**. You must be fluent in using the chain rule to be able to solve related rates problems.

Often in related rates problem you'll have two different rates. The goal will be to find a relationship between them. The best way to learn this is through a few examples. Also, don't forget to always write down equations using algebra and other things to try to determine a relationship between the variables in the problem.

Problem —
What is the rate at which the area of a circle increases when the radius is 4, and the radius increases at a rate of 2 m/s.

Solution: Video Solution

Problem —
There is a cube. The side length of it is increasing at a rate of 4 units. What rate is the volume increasing at when the side length is 3 units?

Solution: Let's assume the side length of the cube is s.
Clearly, $\frac{ds}{dt}$ is 4 since it says the side length is increasing at a rate of 4 units.

In addition to this, we know that $V = s^3$. V represents the volume of the cube.

We want to find the rate of change of V. Thus, we can take the derivative of $V = s^3$ with respect to t
This gives us

$$\frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt}$$

(We can't forget the $\frac{ds}{dt}$ part because of our implicit differentiation techniques!)

Since $\frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt}$, we can plug in our value for $\frac{ds}{dt}$ to simplify the expression. We know that $\frac{ds}{dt}$ is 4. This means that $\frac{dV}{dt} = 12s^2$.

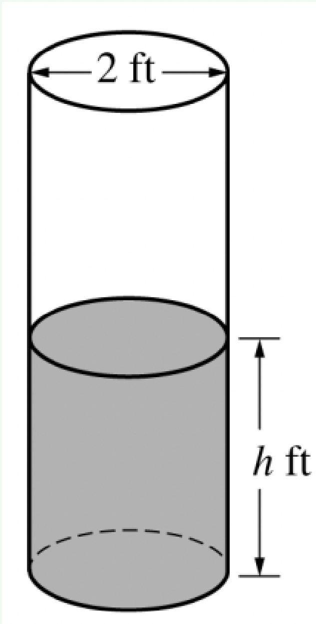
Since we are supposed to find the rate of change of the volume when the side length is 3, we can plug in $s = 3$.

$$\frac{dV}{dt} = 12 \cdot 3^2 = 108$$

§4.5 Solving Related Rates Problems

This section is just about solving related rates problems. There will be 1 example shown below and more later on at the end of this unit.

Problem — 2019 AP Calculus AB FRQ



A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$).

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Solution to part a: The problem is asking us to find $\frac{dV}{dt}$ (rate of change of volume of water).

We already know that if the height of the water is h, then the volume of water $\pi \cdot r^2 \cdot h$. Our value of r (radius) is 1 feet (half of the diameter). We can plug that in to find that the volume of water can be represented as $h\pi$

$V = h\pi$ (equation for volume of water)

We can differentiate both sides with respect to t.

$$\text{That gives us } \frac{dV}{dt} = \pi \cdot \frac{dh}{dt}$$

We are already given the value of $\frac{dh}{dt}$ in the problem statement. It is $-\frac{1}{10}\sqrt{h}$.

We can plug that in to get that $\frac{dV}{dt} = \pi \cdot -\frac{1}{10}\sqrt{h}$

Since we are supposed to find the rate of change of volume of water when the height is 4, we can plug in $h = 4$. This gives that $\frac{dV}{dt} = \pi \cdot -\frac{1}{10} \cdot 2$.

After simplifying, we get $\frac{dV}{dt} = -\frac{\pi}{5}$ cubic feet per seconds.

We will cover more problems later with related rates. It's an important topic, but don't worry. There will be a lot of chances for you to grasp this topic.

§4.6 Approximating Values of a Function Using Local Linearity and Linearization

Some problems will ask you to **approximate** a certain value. You will also have the tangent line to some point. Using that tangent line, you will approximate a certain value.

Let's assume that x_1 is the value we have a tangent line for while x_2 is the value we will plug in to estimate a value. Then, $f(x_2) - f(x_1) = f'(x_1)(x_2 - x_1)$

Note

Is our approximation an overestimate or underestimate?
If our function is **concave up**, then our approximation will be an underestimate. However, if our function is **concave down**, then our approximate will be an overestimate.

The terms concave up and concave down haven't been introduced yet in this book. It's fine if you don't understand this part. It will be explained later.

§4.7 Using L'Hopital's Rule for Determining Limits of Indeterminate Forms

L'Hopital's Rule expands on the content we learned in the limits unit.
Let's pretend we want to find $\lim_{x \to a} \frac{f(x)}{g(x)}$.

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ **or** $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

This means that if the limit of **both** the numerator and denominator is either 0 or ∞ (indeterminate forms), then you can take the derivative of both the numerator and denominator and find the limit of that instead.

Even after we differentiate the numerator and denominator, we might get an indeterminate form again. In that case, apply L'Hopital's Rule again! Differentiate again until you don't have an indeterminate form. This might be confusing still. However, don't worry since examples will be shown soon.

Unit 4 Practice Problems

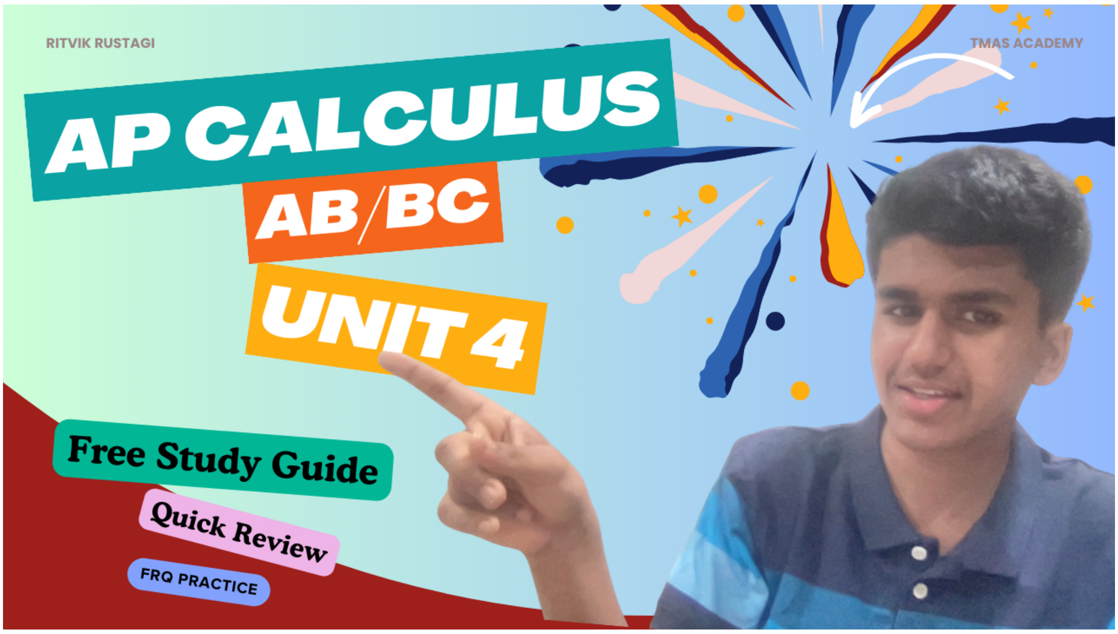
Problem — 1984 AP Calculus AB FRQ
The volume V of a cone is increasing at the rate of 28π cubic inches per second. At the instant when the radius r on the cone is 3 units, its volume is 12π cubic units and the radius is increasing at $\frac{1}{2}$ unit per second.

a. At the instant when the radius of the cone is 3 units, what is the rate of change of the area of its base?

b. At the instant when the radius of the cone is 3 units, what is the rate of change of its height h?

c. At the instant when the radius of the cone is 3 units, what is the instantaneous rate of change of the area of its base with respect to its height h?

Solution: Video Solution



Problem — 2013 AP Calculus AB FRQ
Consider the differential equation $\frac{dy}{dx} = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$

Solution: To write the equation of the tangent line at the point $(1, 0)$, we first need to find the slope.

We are already given an expression for $\frac{dy}{dx}$. Thus, we can just plug in our coordinates to find the slope at that point.

At the point $(1, 0)$, $\frac{dy}{dx} = e^0(3(1)^2 - 6 \cdot 1) = -3$. This means that the slope is -3.

Since the equation of the tangent line can be modeled as $y = mx + b$, the equation simplifies to $y = -3x + b$.

We know that the tangent line goes through the point $(1, 0)$. Thus, we can plug in that point to find the value of b. Doing so gives that $b = 3$.

This means the equation of our tangent line is $y = -3x + 3$
We can use the equation of our tangent line to approximate 1.2

Plugging in 1.2 for x gives $y \approx -3 \cdot 1.2 + 3 = -0.6$

Problem — 2012 AP Calculus AB FRQ
For $0 \leq t \leq 12$, a particle moves along the x-axis. The velocity of the particle after time t is given by $v(t) = \cos(\frac{\pi}{6}t)$. The particle is at position $x = -2$ at time $t = 0$.

For $0 \leq t \leq 12$, when is the particle moving to the left?

Solution: The particle will be moving to the left when $v(t) < 0$. It will be moving to the left when velocity is negative.

We know that cosine is negative in quadrants 2 and 3. This means that it's negative when the angle is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

We can see that at time $t = 3$, $\frac{\pi}{6}t$ will be be $\frac{\pi}{2}$. Similarly, at time $t = 9$, $\frac{\pi}{6}t$ will be be $\frac{3\pi}{2}$.

These are the times that we are looking for.

Problem — 2011 AP Calculus AB FRQ
For $0 \leq t \leq 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by

$$v(t) = 2\sin(e^{\frac{t}{4}}) + 1.$$

The acceleration of the particle is given by

$$a(t) = \frac{1}{2}e^{\frac{t}{4}}\cos(e^{\frac{t}{4}}) \text{ and } x(0) = 2.$$

Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

Solution: The speed will be increasing if the sign of acceleration and velocity are the same (they both can be either negative or positive). However, the speed will be decreasing if the sign of velocity and acceleration are different.

We can use a calculator to plug in $t = 5.5$ into $a(t)$ and $v(t)$.

Doing so gives that $a(t) = -1.36$ and $v(t) = -0.45$.

Since $v(t)$ and $a(t)$ have the same sign at time $t = 5.5$ s, the speed is indeed increasing at that time.

Problem — 1990 AP Calculus AB FRQ
The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second.
(Note: The volume of a sphere with radius r is given by $V = \frac{4}{3}\pi r^3$)

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

Solution to part a: We know that $V = \frac{4}{3}\pi r^3$
We differentiate both sites with respect to t.

Doing so gives $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

We know that $\frac{dr}{dt} = 0.04$. We want to find the rate at which volume increases when the radius is 10 cm.

Plugging in our numbers gives $\frac{dV}{dt} = 4\pi \cdot 10^2 \cdot 0.04 = 16\pi$ cm^3/sec

Solution to part b: We will find the radius at which the volume of the sphere will be 36π cubic centimeters.
Since we know that volume is $V = \frac{4}{3}\pi r^3$, we can equate it to 36π.
$\frac{4}{3}\pi r^3 = 36\pi$

Dividing both sides by π gives $\frac{4}{3}r^3 = 36$
We can isolate r to get $r = 3$.

We know that the cross section that goes through the center will have a radius of r (same radius as the sphere itself).

The area of a circle can be represented through the expression $A = \pi r^2$.

We can differentiate both sides with respect to t to get $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

We can plug in $r = 3$ and $\frac{dr}{dt} = 0.04$ to get that $\frac{dA}{dt} = 2\pi \cdot 3 \cdot 0.04 = 0.24\pi$ cm^2/sec

Problem — 1999 AP Calculus AB FRQ
A particle moves along the y-axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$,
(a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?

(b) Find the acceleration of the particle at time $t = 1.5$ and determine if the velocity of the particle is increasing at $t = 1.5$. Why or why not?

Solution to part a: We can plug in $t = 1.5$ into our expression for velocity. If $v(t)$ is positive for $t = 1.5$, then we will be moving up. However, if it is negative, then we will be moving down.

$v(1.5) = 1.5\sin(1.5^2) = 1.17$

Since $v(t)$ is positive at $t = 1.5$, the particle is moving up.

Solution to part b: We can find acceleration by first differentiating velocity.
Since this is a calculator problem, we can simply use our calculator command for this and plug in $t = 1.5$ both at the same time.

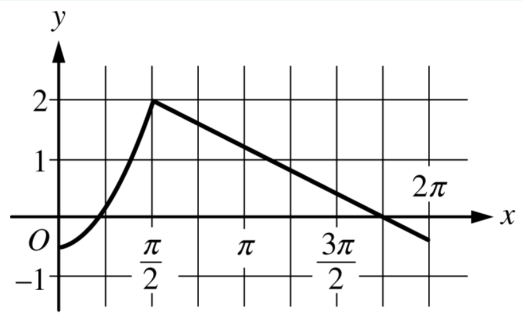
$$a(t) = v'(t) = 2t^2\cos(t^2) + \sin(t^2)$$

Evaluating the above expression at $t = 1.5$ gives that $a(1.5) = -2.05$

The velocity of the particle is decreasing since $a(t) < 0$ at time $t = 1.5$

Problem — 2018 AP Calculus AB FRQ
Let f be the function defined by $f(x) = e^x \cos x$.

Let g be a differentiable function such that $g(\frac{\pi}{2}) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

Solution: To find $\lim_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)}$, I will first find $\lim_{x\to\frac{\pi}{2}} f(x)$ and $\lim_{x\to\frac{\pi}{2}} g(x)$.

$$\lim_{x\to\frac{\pi}{2}} f(x) = f(\frac{\pi}{2}) = e^{\frac{\pi}{2}}\cos(\frac{\pi}{2}) = 0$$

$$\lim_{x\to\frac{\pi}{2}} g(x) = g(\frac{\pi}{2}) = 0$$

Since both the numerator and denominator head towards 0 for their limits, we have an indeterminate form $\frac{0}{0}$. Thus, we can apply L' Hospital's Rule.

$$\lim_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)}$$

We will individually find $\lim_{x\to\frac{\pi}{2}} f'(x)$ and $\lim_{x\to\frac{\pi}{2}} g'(x)$.

Before finding $\lim_{x\to\frac{\pi}{2}} f'(x)$, we need to calculate $f'(x)$.

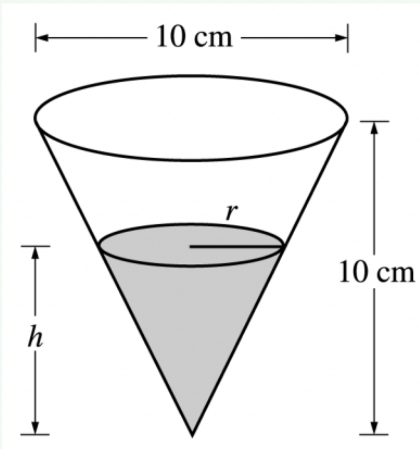
Using the product rule, we can find that $f'(x) = e^x(\cos(x) - \sin(x))$

This means I have to evaluate $\lim_{x\to\frac{\pi}{2}} e^x(\cos(x) - \sin(x))$. This can be done through direct substitution which gives $e^{\frac{\pi}{2}}(\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2}))$. This simplifies to $-e^{\frac{\pi}{2}}$.

Now, I can also find $\lim_{x\to\frac{\pi}{2}} g'(x)$. We already know that $g'(\frac{\pi}{2})$ is 2 due to the graph given. Thus, the limit is also 2 due to direct substitution.

$$\text{This means } \lim_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)} = -\frac{e^{\frac{\pi}{2}}}{2}$$

Problem — 2002 AP Calculus AB FRQ



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $-\frac{3}{10}$ cm/hr.
(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

Solution to part a: We know that the volume is $V = \frac{1}{3}\pi r^2 h$
We know that the cone in the diagram has a height of 10 with a diameter of 10. A diameter of 10 means that it has a radius of 5. Due to similar triangles, we know that all cones inside this larger cone will have a radius that is half the height.

Thus, for a cone with height h, the radius will be equivalent to $\frac{h}{2}$.
This means that the volume can be written as

$$V = \frac{1}{3}\pi(\frac{h}{2})^2 \cdot h = \frac{\pi h^3}{12}$$

Since our height is 5, we can plug that in to get that the volume is $V = \frac{\pi 5^3}{12} = 32.72cm^3$

Solution to part b: We have 2 main variables in this problem. One is volume V while the other is height h. Thus, we should try to relate the two variables together. In part a, we already volume in terms of height.

$$V = \frac{\pi h^3}{12}$$

Now, we should relate our topic of chain rule to related rates.

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

The left side is what we want to find. It is the rate of change of volume.
We should first find $\frac{dV}{dh}$.

$$\frac{dV}{dh} = \frac{d}{dh}[\frac{\pi h^3}{12}] = \frac{\pi h^2}{4}$$

We also already know that $\frac{dh}{dt} = -\frac{3}{10}$

We can plug these values in to find that $\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot -\frac{3}{10} = -\frac{3\pi h^2}{40}$

Since we want to find the rate of change of volume when the height is 5, we can simply plug in $h = 5$.

$$\text{Doing so gives } \frac{dV}{dt} = -\frac{3\pi(5)^2}{40} = -\frac{15\pi}{8} \text{ cm}^3/\text{hr}$$

Problem — 2004 AP Calculus AB FRQ
A particle moves along the y-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$.

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

Solution to part a: We will differentiate velocity to find acceleration.

$$a(t) = \frac{d}{dt}[v(t)] = \frac{d}{dt}[1 - \tan^{-1}(e^t)] = -\frac{e^t}{1 + (e^t)^2} = -\frac{e^t}{1 + e^{2t}}$$

For those who are confused on how to differentiate $\tan^{-1}(e^t)$, think about the chain rule.

Our outer function is $\tan^{-1}(t)$ while the inner one is e^t.

Now that we have an expression for acceleration, we can plug in $t = 2$ to find the acceleration at that time.

$$a(2) = -\frac{e^2}{1+e^{2\cdot 2}} = -\frac{e^2}{1+e^4} = -0.133$$

Solution to part b: The speed will be increasing at a certain time if both acceleration and velocity have the same sign at that time. However, speed will be decreasing if acceleration and velocity have different signs.

We must find the sign of $a(2)$ and $v(2)$.

We can plug in 2 in $v(t) = 1 - \tan^{-1}(e^t)$ to get $v(2) = 1 - \tan^{-1}(e^2) = -0.436$

We already know that $a(2) = -0.133$ from part a.

Since $a(t)$ and $v(t)$ have the same sign at $t = 2$ s, our speed is indeed increasing at that time.

Problem — 2007 AP Calculus BC FRQ

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 \leq t \leq 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \leq t \leq 12$. The radius of the balloon is 30 feet when $t = 5$.
(Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$)

Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

Solution: We know that the volume of a sphere is $V = \frac{4}{3}\pi r^3$

We can differentiate both sides with respect to r to get $\frac{dV}{dr} = 4\pi r^2 \cdot \frac{dr}{dt}$

Don't forget to include the $\frac{dr}{dt}$ part from implicit differentiation.

At time $t = 5$, we know that $r'(t) = 2.0$ (as given in the table).
We also know that $r = 30$ at time $t = 5$.

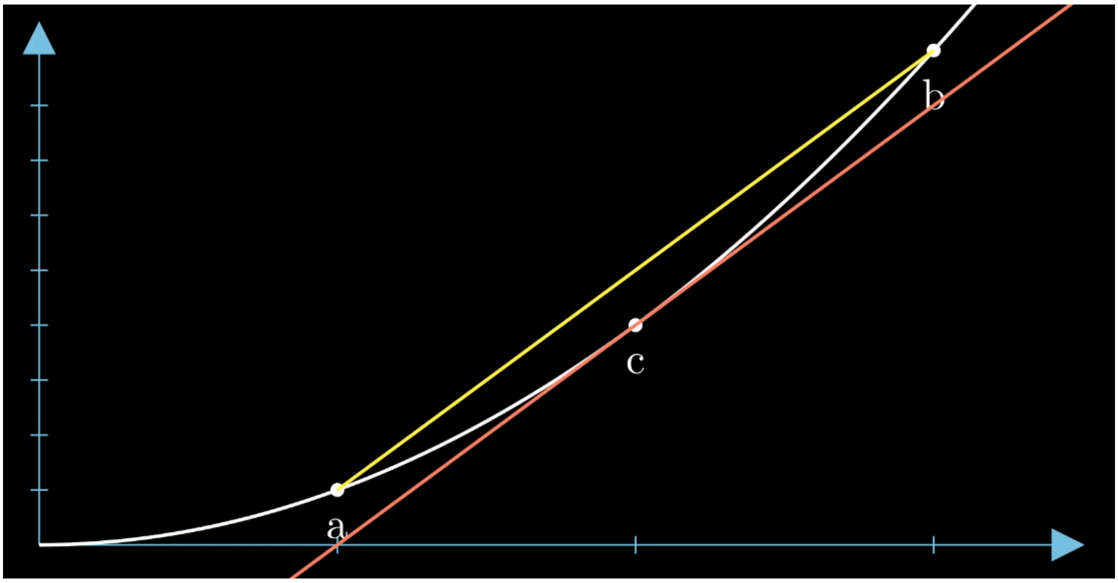
We can plug our known values in to get that $\frac{dV}{dr} = 4\pi \cdot (30)^2 \cdot 2 = 7200\pi$ ft^3/min

Unit 5 Analytical Applications of Differentiation

§5.1 Using the Mean Value Theorem

The Mean Value Theorem is often used to prove a certain statement regarding an interval for a function.

If f is continuous for the closed interval $[a, b]$, and f is differentiable on (a, b), then there exists a point c in the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



In the diagram above, the yellow line represents the secant line which basically represents the average rate of change between points a and b. The red line represents the tangent line to some point c. Based on the Mean Value Theorem, there will be a point c such that the slope of the tangent line to that point will be equivalent to the slope of between a and b (the secant line).

Let's also review the intermediate theorem since it commonly shows up with problems from this unit.

Theorem 5.0.1

Intermediate Value Theorem
The Intermediate Value theorem conditions are that a function f must be continuous on an interval $[a, b]$. Then, there will be a number N in the interval $(f(a), f(b))$ (between the two endpoints) for which a number c exists in (a, b) such that $f(c) = N$

These two theorems are often used to verify certain conditions on the AP exam for calculus. It's really important to know both and when to apply them.

§5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

Theorem 5.0.1

Extreme Value Theorem
The extreme value theorem's condition is that a function f must be continuous for the closed interval $[a, b]$. If that is satisfied, then there will be a minimum and maximum value in the interval. The maximum value will be known as the global maximum while the minimum value is known as the global minimum.

While working with the extreme value theorem, it is important to consider the difference between global extreme and relative extrema.

Global extrema are the smallest/largest outputs of a certain function when taken as a whole. On the other hand, relative extrema are the smallest/largest values of a part of the function.

For an interval, there can be multiple relative minima and maxima, but there can only be one global minimum and one global maxmimum. This might be confusing right now. However, examples will be shown soon to clarify it.

Note

How to Find Critical Points
It's crucial to know how to find critical points for a function. It will allow you to soon find relative extrema.

To find the critical points for a certain function $f(x)$, you first take the derivative of it. Then, you set the derivative of the function $f(x)$ which is $f'(x)$ to 0. You solve for values of x that make the function 0. All of those values count as critical points.

On top of that, you find values for which the derivative **does not exist** or is undefined. This part is super important. Any value of x for which the derivative is undefined is also a critical point.

Remember that all local extrema occur at critical points. That means to find local extrema, we must first find critical points. However, also keep in mind that not all critical points will also be a local extrema. That is why we must test each critical point.

§5.3 Determining Intervals on Which a Function Is Increasing or Decreasing

Now using critical points, how would you determine for what interval our function is **increasing** or **decreasing**?

Just as a reminder, a function is increasing at points for which the **first derivative is positive**. However, a function is decreasing at points for which the **first derivative is negative**.

Critical points are used to find the intervals for which a function is increasing/decreasing.

For all of the critical points that are listed, you will test values between those critical points and plug them into $f'(x)$ (first derivative equation). If the value is negative, then the function is decreasing between those critical points. However, if that value is positive, then the function is increasing between those critical points.

I highly recommend learning the number line trick to easily find the intervals where a function decreases or increases. You can learn about it in the AP Calculus AB/BC Unit 5 Rapid Review video on the TMAS Academy youtube channel.

Problem — $f(x) = \frac{x^2-6}{2x+5}$
Find the intervals where $f(x)$ is decreasing and increasing. Give justification for why.

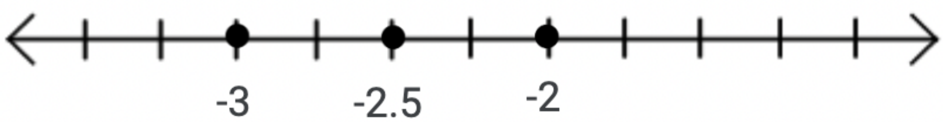
Solution: In this problem, we first find the derivative of our function. This will allow us to find the critical points.

Using the quotient rule, we get that $f'(x) = \frac{2x^2+10x+12}{(2x+5)^2}$
We can factor $f'(x)$ to write it as

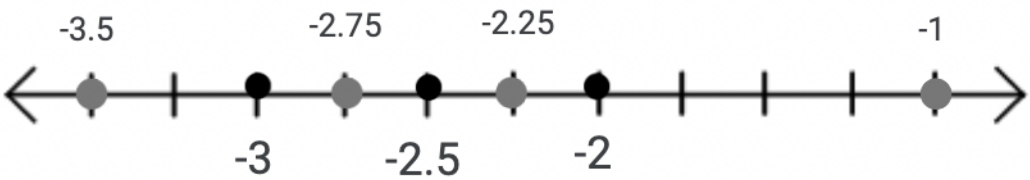
$$\frac{2(x+2)(x+3)}{(2x+5)^2}$$

From here, we equate our expression for $f'(x)$ to 0. This gives that $x = -2$ and -3 are critical points.
To find the points for which $f'(x)$ are undefined, we equate the denominator to 0. Solving $(2x+5)^2 = 0$ gives that $x = -\frac{5}{2}$ is also a critical point.

Now, I recommend making a number line labelling the critical points in order from least to greatest.

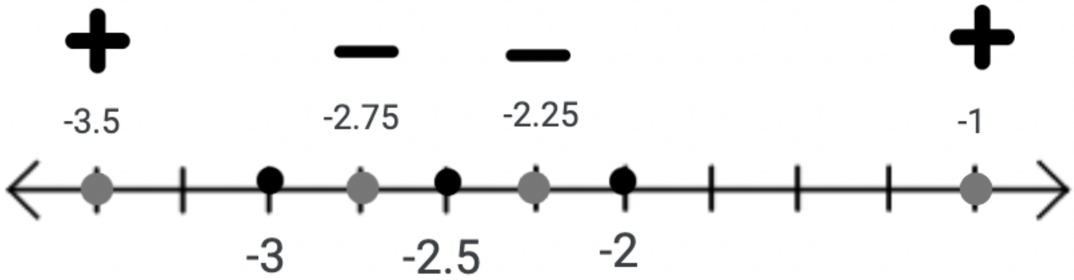


Now choose a value between each critical point. Choose an easy value that is easy to test. Don't choose something like -2.6667.
Don't forget to choose a number at the left of the leftmost endpoint and at the right of the rightmost endpoint.



Now, we plug in each of the values we chose into $f'(x)$. If the value for $f'(x)$ is negative for that point, then label a negative sign on top of that number. If it's positive, then label a positive sign on top of that number.
The numbers we are testing are $-3.5, -2.75, -2.25$, and -1.

We get that $f'(-3.5)$ and $f'(-1)$ are positive while $f'(-2.75)$ and $f'(-2.25)$ are negative.



The sign for $f'(x)$ at that certain break-point applies for all the points in that category. For $x < -3$ and $x > -1$, $f(x)$ will be increasing because $f'(x)$ is positive. However, for $-3 < x < -2.5$ and $-2.5 < x < -2$, $f(x)$ will be decreasing because $f'(x)$ is negative.

§5.4 Using the First Derivative Test to Determine Relative (Local) Extrema

Note

The First Derivative Test
The First Derivative Test allows us to find the relative (local) extrema. To find the relative extrema, you first find the critical points. Then, you check to see the sign of the derivative on the left and right side of each critical point.

If the sign of the derivative changes from negative to positive, then we have a relative minimum. However, if the sign of the derivative changes from positive to negative about the critical point, then we have a relative maximum.

Problem — For the function $f(x) = \frac{x^3}{x+2}$, find the relative extrema.

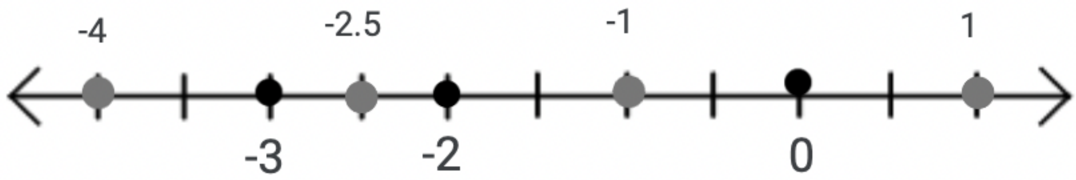
Solution: We first find $f'(x)$ to be able to find the critical points.
$f'(x)$ can be found using the quotient rule, and it is $\frac{x^2(2x+6)}{(x+2)^2}$

Equating this to 0 gives that the critical points are 0 and -3.

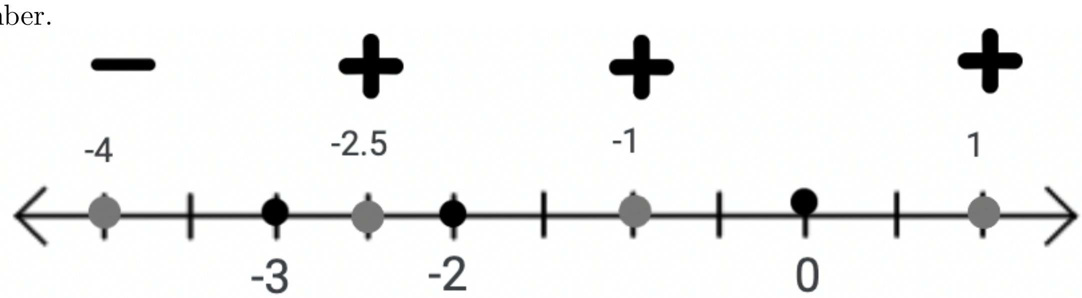
$f'(x)$ is undefined when the denominator is 0. This gives that another critical point is

$x = -2$.
Thus, the three critical points are $-3, -2$, and 0.

We plot all 3 points on a number-line. Then, we choose a value in between each critical point to test. We will also choose a value to the left of our leftmost critical point along with a value to the right of our rightmost critical point.



The black dots represent the critical points while the grey dots represent the values we will test.
We find the sign of $f'(x)$ for $-4, -2.5, -1$, and 1. We'll label the sign on top of that number.



We know that a relative extrema occurs when the sign of $f'(x)$ changes. That only happens for the critical point **-3**. We have a relative minimum there since the first derivative goes from negative to positive.

§5.5 Using the Candidates Test to Determine Absolute (Global) Extrema

Candidate's Test
It's important to know the candidate's test if you want to find the **absolute extrema** of a certain function for a certain interval.

If we are given a function $f(x)$ and we are asked to find the absolute extrema for the interval $[a, b]$, then we first find the critical points for $f(x)$. Then, we'll test each critical point and endpoint (in this case a and b). The maximum out of all these values will be our absolute maximum for this interval while the minimum will be our absolute minimum.

> **Problem** — $f(x) = x^3 + x^2 - 8x$
> Find the absolute extrema for $f(x)$ for the interval $[-1, 4]$

Solution: First, we find the critical points for $f(x)$.
We simply take the derivative and find the values that make $f'(x)$ either 0 or undefined.
Using the power rule, we get that $f'(x) = 3x^2 + 2x - 8$

We can factor our expression for $f'(x)$ to get $(3x-4)(x+2)=0$

Solving this gives that our critical points are $x=\frac{4}{3}$ and -2.

This means the values that we test will be $\frac{4}{3}$ and -2. However, we must also test the endpoints of the interval which are -1 and 4.

We plug all these values into $f(x)$. The largest value will be the absolute maximum while the smallest one will be the absolute minimum.

$f(-2)=12$
$f(-1)=8$
$f(\frac{4}{3})=-\frac{176}{27}$
$f(4)=48$

Clearly our absolute maximum occurs at $x=4$ while our absolute minimum occurs at $x=\frac{4}{3}$.

§5.6 Determining Concavity of Functions over Their Domains

Theorem 5.0.1

A graph is **concave up** for all points where $f''(x)>0$ (the second derivative is more than 0). A graph is **concave down** for all points where $f''(x)<0$ (the second derivative is less than 0.

Note

How to find the intervals of concavity?
First, you find $f''(x)$. Then, you equate it to 0 and solve for the solutions.
After that, you make the number-line that we made for our first derivative test. That number-line will be used to determine concavity direction (concave up or down)

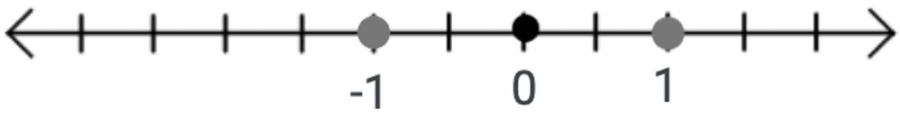
Also, remember that a **point of inflection** occurs when the sign of $f''(x)$ changes!

Problem — What are the intervals of concavity for $f(x)=6x^3-18x$?

Solution: The first step is to find $f''(x)$ (the second derivative).
$f'(x)=18x^2-18$
$f''(x)=36x$

Equating $36x$ to 0 gives us that the only value we need to test is $x=0$.

On a number-line, I can label 0. Then, I choose a value to the left of 0 to test and one value to the right of 0. We'll test -1 and 1.



Now, we plug in -1 and 1 into $f''(x)$. If the sign of $f''(x)$ is negative, then it will be concave down for that interval. If the sign is positive, then it will be concave up for that interval.

Clearly, the graph is concave up for $(0, \infty)$ while it is concave down for $(-\infty, 0)$.

Remember that a function $f(x)$ will have a point of inflection at the point where the concavity changes. In this case, there is a point of inflection at 0 since the function is concave down to the left of 0 but concave up to the right of 0.

§5.7 Using the Second Derivative Test to Determine Extrema

Note

The Second Derivative Test
The Second Derivative Test allows us to find the relative (local) extrema. To find the relative extrema using this method, you first find all points c for which $f'(c) = 0$.

Then, you plug in all of those points into $f''(x)$. If $f''(c) > 0$, then our function f will have a relative minimum at $x = c$.

However, if $f''(c) < 0$, then our function f will have a relative maximum at $x = c$.

In an FRQ that involves the second derivative test for a certain function $f(x)$, make sure to give a reasoning after.
For example, if you find out that there is a relative minimum at $x = c$, then make sure to say: there is a relative minimum at $x = c$ because $f'(c) = 0$ and $f''(c) > 0$.

On top of that, if you want to say that there is a relative maximum at $x = c$, then make sure to say: there is a relative maximum at $x = c$ because $f'(c) = 0$ and $f''(c) < 0$.

Problem — Find the relative extrema using the Second Derivative Test for $f(x) = 2x^3 - 3x^2$

Solution to part a: We will first find all points c for which $f'(c) = 0$.
Using the power rule, we know that $f'(x) = 6x^2 - 6x$
Setting this equal to 0 gives that possible solutions are 0 and 1.

Now we find the second derivative which is $f''(x) = 12x - 6$

We can plug in $x = 0$ to get $f''(0) = 12 \cdot 0 - 6 = -6$

Since $f''(x) < 0$ for $x = 0$, we have a relative maximum at $x = 0$.

We do the same thing for $x = 1$.

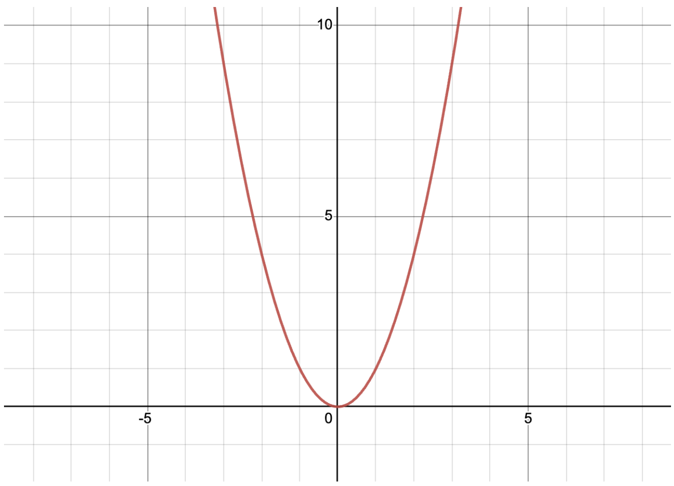
Plugging it into $f''(x)$ gives that $f''(1) = 12 \cdot 1 - 6 = 6$

Since $f''(x) > 0$ for $x = 1$, we have a relative minimum at $x = 1$.

§5.8 Sketching Graphs of Functions and Their Derivatives

How would you sketch the graph of the derivative function?
The most important part is determining the sign of the derivative and making an approximate model based on the parent function.

For example, let's pretend $f(x) = x^2$. If we want to graph the derivative function without directly computing the derivative, we'll have to investigate with the signs.



The graph of $f(x) = x^2$ is given above. It's obvious that the slope for all points $x < 0$ is negative. However, for all points $x > 0$, the slope is positive.

That means the graph for $f'(x)$ will be above the x-axis for all points $x > 0$. However, $f'(x)$ will be below the axis for all points $x < 0$.

Some important **tips** are to first label the points that have a derivative of 0 for $f(x)$. For example, $f(x) = x^2$ has a derivative of 0 at $x = 0$. This means we can immediately graph $0, 0$ for the graph of $f'(x)$. Such techniques can make it easy for us to draw the graph.

For the most part, you most likely won't have to graph the derivative function for an FRQ. You can see such a problem for a multiple-choice question. The graph of $f(x)$ will be given. There will be a few options showing graphs of $f'(x)$. The best way to solve such problems is to process of eliminate by checking the points where the derivative is 0 and where it's positive/negative.

§5.9 Connecting a Function, Its First Derivative, and Its Second Derivative

Remember when speed is increasing/decreasing.
An object's **speed** will be increasing if acceleration and velocity have the same sign. However, the **speed** will be decreasing if acceleration and velocity have different signs.

Remember how a function and its first derivative and second derivative relate to each other. Remember what the sign of the first and second derivative can convey about the function.

Go back and review when a function is concave up/down. You also need to know how to find out if a function is increasing/decreasing.

You still might be confused on this topic. Don't worry since the practice problems for this unit will help.

§5.10 and 5.11 Optimization

For optimization problems, you must write an equation to represent the problem. You will often be given a word problem. However, you must convert it to mathematical equations.

You should also draw a diagram to represent the problem. Use variables whenever you can to represent important factors in the problem such as side length.

You will often have to take the first derivative to find minimum/maximum values. Don't forget about critical points! The best way to learn this topic is through examples.

Problem —
We will make an open-top box by cutting out squares of length s from each corner of a rectangle with side lengths of 40 cm and 50 cm. After cutting out squares from each corner, we will fold up the sides. What should the side length s of the cut-out squares be so that the volume is maximized?

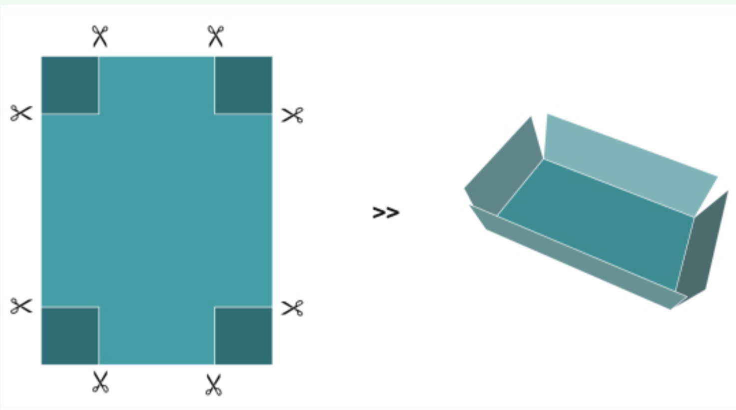


Image Credits: Mathigon

Solution: The height of our open-top box will be s (the side length of the cut-out squares). The lengths of the base of our box will be $40 - 2s$ and $50 - 2s$. The reason is that there are two squares that are cutout from each side of our original rectangle.

The part above is the toughest to realize. Make sure to label all of the side lengths to be able to understand this. If you're still confused, then don't forget to join the TMAS Academy Discord server, the place where you can get all the help necessary.

We were able to find out that the box has side lengths of s, $40 - 2s$, and $50 - 2s$.
We know that the volume of a rectangular prism is lwh. We simply multiply all 3 sides,

For this problem, $V(s) = s(40 - 2s)(50 - 2s) = 2000s - 180s^2 + 4s^3$

We want to find the maximum value of $V(s)$. Also, remember that $s < \frac{40}{2} = 20$. If it's more than 20, then two combined squares will be longer than the side length of our rectangle which isn't possible.

We must find the critical points for $V(s)$.
To do that, we find the first derivative of $V(s)$.

From the power rule, we know that $V'(s) = 2000 - 360s + 12s^2$

We can use the quadratic formula or a calculator to find that the solutions to the equation are $s = 7.36$ and $s = 22.64$

Since we know that s must be less than 20, our solution is $s = 7.36$.
We can even plug in a number less than 7.36 to see that $V'(s) > 0$ for any such value. That means that volume is increasing till that point where it hits a relative maximum when $s = 7.36$

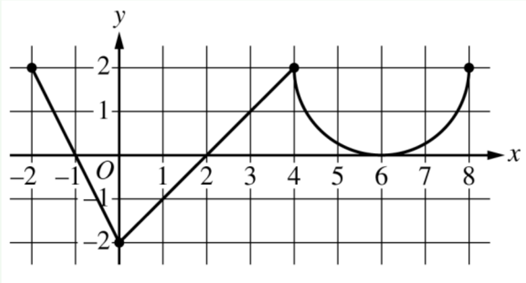
This is indeed very confusing for many people. There will be more examples that you will see related to optimization soon.

§5.12 Exploring Behaviors of Implicit Relations

Problems from this section will often involve implicit differentiation.
Be consistent with the rules you learned. For example, a critical point will still exist when $\frac{dy}{dx} = 0$ or is undefined even for implicit relations. Those fundamental ideas will be the same! It's just that the method to find $\frac{dy}{dx}$ might be different from other problems.

Unit 5 Practice Problems

Problem — 2023 AP Calculus BC FRQ



Graph of f'

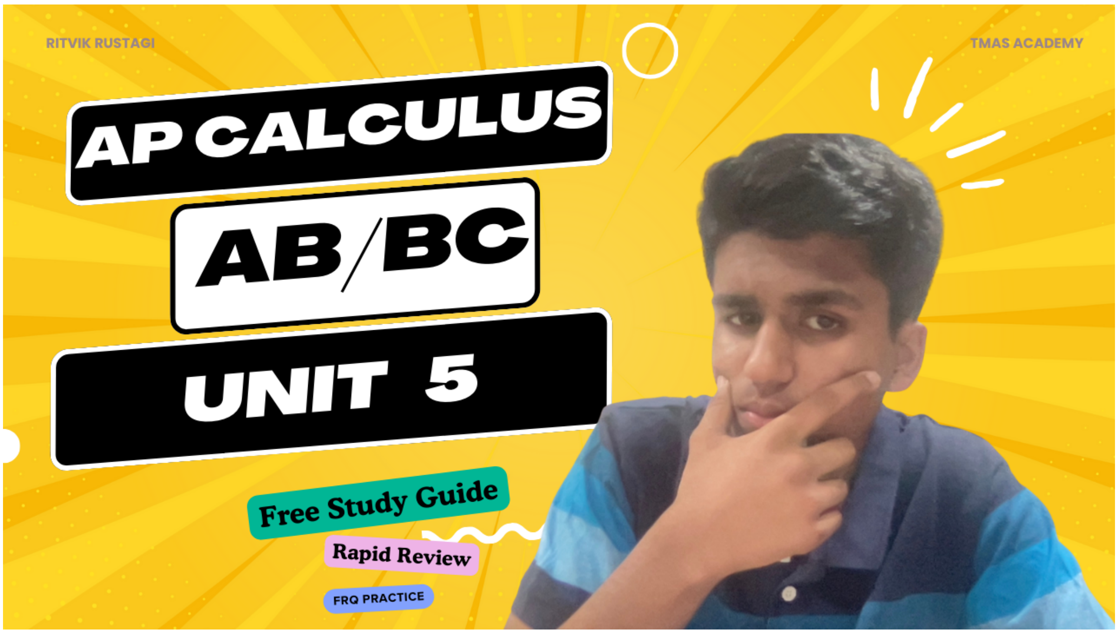
The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$. Give a reason for your answer.

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6}$, or show that it does not exist. Justify your answer.

Solution: Video Solution



Problem — 2018 AP Calculus BC FRQ

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

Solution to part a: Since 6 lies between 5 and 7 in the table, we can estimate $H'(6)$ by using $t = 5$ and $t = 7$. Our estimation will be $\frac{H(7)-H(5)}{7-5}$ which is 2.5.

To interpret it, you can say, $H'(6) = 2.5$ is the rate at which the height of the tree increases each year at $t = 6$ years.

Solution to part b: Whenever a problem asks you to explain why a certain value exists, then you should immediately think about something like Mean Value Theorem.

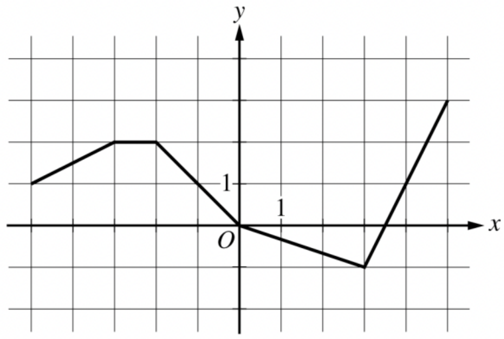
Before applying Mean-Value Theorem, we MUST check its conditions.

First of all, we notice that $\frac{H(5)-H(3)}{5-3} = 2$. This means that there exists a value c satisfying $3 < c < 5$ such that $H'(c) = 2$ if the conditions are satisfied.

Clearly the conditions are satisfied because $H(t)$ is continuous and differentiable on $3 \leq t \leq 5$.

Problem — 2017 AP Calculus AB FRQ

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h

Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

Solution to part a: For this part, we simply differentiate $f(x)$. Then, we'll plug in π into $f'(x)$.

$$f'(x) = -2\sin(2x) + \cos(x)e^{\sin x}$$

We now plug in π to get $f'(\pi) = -2\sin(2\pi) + \cos(\pi)e^{\sin \pi}$. This evaluates to **-1** which is the slope.

Solution to part b: To find the derivative of $k(x)$, we must use the chain rule because we have a nested function.

Since $k(x) = h(f(x))$, $k'(x) = h'(f(x)) \cdot f'(x)$.

We plug in π to get that $k'(\pi) = h'(f(\pi)) \cdot f'(\pi)$.

To evaluate $h'(f(\pi))$, we first evaluate $f(\pi)$. We plug this into our expression for $f(x)$ to get that $f(\pi)$ is $cos(2\pi) + e^{\sin(\pi)} = 2$.

This means that $h'(f(\pi)) = h'(2)$. We can find $h'(2)$ by observing our graph of h.

We can tell that the slope of $x = 2$ is simply $-\frac{1}{3}$.

Now, we can get $f'(\pi)$ by simply using our answer from part a which was -1.

This means $k(\pi)$ is $-\frac{1}{3} \cdot -1$ which is $\frac{1}{3}$.

Solution to part c: We can find the derivative of $m(x)$ by using the product rule.

Using the product rule gives that

$$m'(x) = -2g'(-2x) \cdot h(x) + g(-2x) \cdot h'(x)$$

We can plug in $x = 2$ into this to get that

$$m'(2) = -2g'(-4) \cdot h(2) + g(-4) \cdot h'(2)$$

By observing the table for values related to function g, we find that $g'(-4) = -1$ and $g(-4) = 5$.

We can find $h(2)$ by observing the graph which gives that $h(2) = -\frac{2}{3}$.
Similarly, we can find that $h'(2)$ is $-\frac{1}{3}$.

Now we can plug in these values into the expression we found: $m'(2) = -2g'(-4) \cdot h(2) + g(-4) \cdot h'(2)$

We get $m'(2) = -2 \cdot -1 \cdot -\frac{2}{3} + 5 \cdot -\frac{1}{3}$ which is **-3**.

Solution to part d: This problem makes it evident that it involves the Mean Value Theorem. The reason is that it involves an interval and asks us to confirm if a value exists in the interval for which a certain derivative exists.

First of all, we will prove that the Mean Value Theorem conditions are satisfied. They clearly are because our function $g(x)$ is differentiable and it is also continuous on the interval $[-5, -3]$.

This means that if c is between -5 and -3, then there must be a value for which $g'(c) = \frac{g(-3)-g(-5)}{-3-(-5)} = \frac{2-10}{2} = -4$. We have clearly verified that a value c exists for which $g'(c) = -4$ by using the Mean Value Theorem.

Problem — 2014 AP Calculus AB FRQ

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value for c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

Solution to part a: A relative minimum occurs when the first derivative changes signs from negative to be positive. Clearly at $x = 1$, we can see in the table that $f'(x)$ goes from negative to positive. Thus, the answer is $x = 1$.

Solution to part b: We can apply the mean value theorem on $f'(x)$ over the interval $[-1, 1]$. We notice that our conditions of $f'(x)$ being continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ are met for the interval. Thus, we can apply Mean Value Theorem.

Applying the Mean Value Theorem gives $f''(c) = \frac{f'(1)-f'(-1)}{1-(-1)} = 0$.

This means that a value c such that $-1 < c < 1$ and $f''(c) = 0$ must exist due to the Mean Value Theorem.

Solution to part c: In this problem, we must apply the chain rule since we have a nested function. The outer function is $\ln x$ while the inner function is $f(x)$.

Doing so gives that $h'(x) = \frac{1}{f(x)} \cdot f'(x)$

Now we can plug in 3 into this to find that $h'(3) = \frac{1}{f(3)} \cdot f'(3)$.
From the table, we can observe that $f(3) = 7$ while $f'(3) = \frac{1}{2}$.

Plugging these values into the expression for $h'(3)$ gives us an answer of $\frac{1}{14}$.

Problem — 2001 AP Calculus AB FRQ
Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \frac{x^2-2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

Solution to part a: There will be a horizontal tangent when $h'(x) = 0$
We can set our expression for $h'(x)$ to 0 and solve.
We solve $\frac{x^2-2}{x} = 0$

We can solve to find that $x = \pm\sqrt{2}$. $h(x)$ will have horizontal tangents at $x = \pm\sqrt{2}$.

We will now use our number line trick to investigate if $h'(x)$ is positive/negative around our critical values. We already found 2 of them, and they are $x = \pm\sqrt{2}$. We must also find the values for which $h'(x)$ is negative. That will occur when the denominator is 0. That means we must also test $x = 0$.

That means our 3 numbers to mark on the number-line are $-\sqrt{2}$, 0, and $\sqrt{2}$.

We can plug in a number less than $-\sqrt{2}$ to see if $h'(x)$ will be positive or negative. We can try -2. Plugging it in gives $h'(2) = \frac{(-2)^2-2}{-2} = -1$. Clearly, $h'(x) < 0$ when $x < -\sqrt{2}$.

Now, we can test a number between $-\sqrt{2}$ and 0. We can test -1. Doing so gives $h'(-1) = \frac{(-1)^2-2}{-1} = 1$. Clearly, $h'(x)$ is positive between $-\sqrt{2}$ and 0.

Now, we will test a number between 0 and $\sqrt{2}$. We will test 1. We get $h'(1) = \frac{1^2-2}{1} = -1$. This means that $h'(x)$ will be negative between 0 and $\sqrt{2}$.

We can test a value above $\sqrt{2}$. We will test 2. We get $h'(2) = \frac{2^2-2}{2} = 1$. This means that $h'(x)$ will be positive when $x > \sqrt{2}$

Clearly, sign change occurs at $-\sqrt{2}$, 0, and $\sqrt{2}$. We must ignore 0 since it isn't part of the domain for $h(x)$.

We can conclude that there are local maxima at $x = \sqrt{2}$ and $x = -\sqrt{2}$ since the sign of $h'(x)$ changes from negative to positive.

Solution to part b: We know that concavity involves the second derivative. We can differentiate $h'(x)$ using the quotient rule.

$$\text{Doing so gives } h''(x) = \frac{x^2+2}{x^2}$$

Now, we must find inflection points by setting the expression for $h''(x)$ to 0.

We get the equation $\frac{x^2+2}{x^2} = 0$. We only need to test $x = 0$.

We can test a value less than 0. For example, if we test -1, then we get $h''(-1) = \frac{(-1)^2+2}{(-1)^2} = 3$.
Now, we can also test a value above 0. If we test 1, then we get $h''(1) = \frac{1^2+2}{1^2} = 3$

Clearly, $h(x)$ is concave up on the intervals $(-\infty, 0)$ and $(0, \infty)$ because $h''(x) > 0$ for both intervals.

Solution to part c: We find the slope at $x = 4$ by plugging in $x = 4$ into $h'(x)$.

$$\text{Doing so gives } h'(4) = \frac{4^2-2}{4} = \frac{7}{2}.$$

We can plug the slope into $y = mx + b$ to get $y = \frac{7}{2}x + b$
We know that the tangent line intersects the curve at $(4, -3)$.

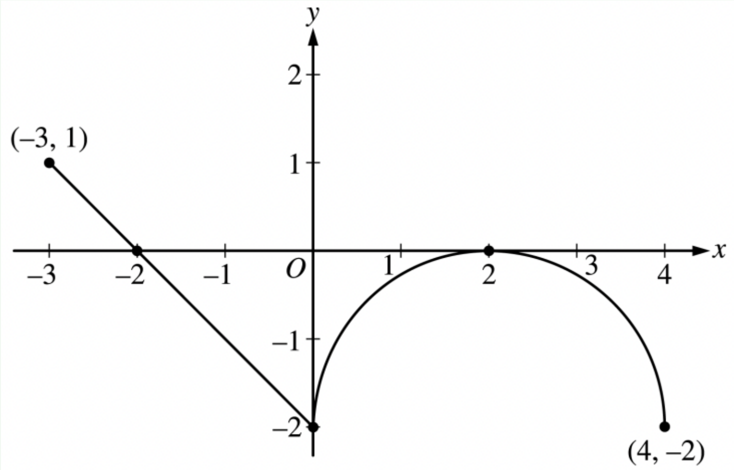
Thus, we can plug this point into our equation for the tangent line.
Doing so gives $-3 = \frac{7}{2} \cdot 4 + b$
We can solve for b to get $b = -17$.

This means that the equation for the tangent line is $y = \frac{7}{2}x - 17$

Solution to part d: The tangent line will lie below a point if the function is concave up at that point. On the other hand, it will lie above a function if the function is concave down at that point.

Our function $h(x)$ is concave up at the points $x > 4$. Thus, our line tangent to the graph at $x = 4$ lies below the graph of h.

Problem — 2003 AP Calculus AB FRQ



Graph of f'

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

Solution to part a: $f(x)$ is increasing whenever $f'(x)$ is positive.
We will observe the given graph. Clearly, the range of values above the x-axis (meaning they are positive) occurs when $-3 < x < -2$.

Solution to part b: We will find values of x when $f''(x) = 0$ or is undefined.
That occurs at $x = 0, 2$.

To the left of 0, $f''(x) < 0$ because $f'(x)$ is decreasing. Between 0 and 2, $f''(x) > 0$ because $f'(x)$ is increasing. This means that there is a point of inflection at $x = 0$ since the sign of $f''(x)$ changes at $x = 0$.

We know that between 0 and 2, $f''(x) > 0$ because $f'(x)$ is increasing. We will now observe what happens to $f''(x)$ to the right of 2. Clearly, $f''(x) < 0$ since $f'(x)$ is decreasing. This means there is another point of infection at $x = 2$ since the sign of $f''(x)$ changes at $x = 2$.

Solution to part c: At $x = 0$, we can observe the graph to see that $f'(0) = -2$.
This means that the slope of the tangent line is -2 at $x = 0$.
We can plug in -2 for the slope in the equation $y = mx + b$.
Doing so gives $y = -2x + b$

We know that the tangent line goes through the point $(0, 3)$.

We can plug that point in to get $3 = -2 \cdot 0 + b$ which gives that $b = 3$.

That means the equation of the tangent line at the point $(0, 3)$ is $y = -2x + 3$.

Problem — 2005 AP Calculus AB FRQ

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	–1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	–3	Negative
$f''(x)$	–2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at 2.

For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

Solution: To find the relative extrema, we need find the points for which there is a sign change for $f'(x)$.
There will be a relative minimum at $x = 0$ since that's the start of the interval and also the point where $f(x)$ starts to increase.

There is a relative maximum at $x = 2$ since $f'(x)$ changes from positive to negative.

Problem — 2007 AP Calculus AB FRQ

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

Solution to part a: We plug in $x = 3$ into $h(x)$.

$$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$$

We plug in $x = 1$ to get $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

-5 lies between $h(1)$ and $h(3)$ and h is continuous. Thus, we can use the Intermediate Value Theorem (since our conditions are satisfied), and we know that there exists a value r for $1 < r < 3$ such that $h(r) = -5$.

Solution to part b: This problem speaks the Mean Value Theorem since it's asking us to confirm if a certain value exists for $h'(c)$.

We can find the average rate of change (slope of the secant line) between $x = 3$ and $x = 1$.
Doing so gives $\frac{h(3)-h(1)}{3-1} = \frac{-7-3}{2} = -5$.

Since h is continuous and differentiable, a value c for $1 < c < 3$ will indeed exist such that $h'(c) = -5$, and this is backed up by the Mean Value Theorem.

Problem — 2009 AP Calculus AB FRQ
The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \leq t \leq 2$; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

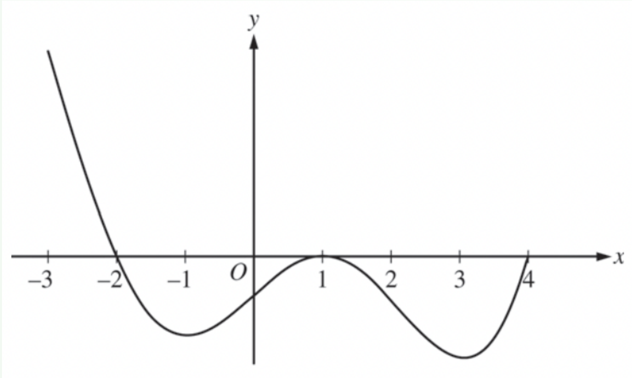
Solution: We can find the maximum by working with $R'(t)$. We must find the critical points.
Since this is a calculator problem, we can use our calculator to find times t for which $R'(t) = 0$.

Doing so gives that $t = 0$ and $t = 1.363$
Also, we must test $t = 2$ since we have a closed interval which means that **we must test our endpoints**!
We can use our calculator to find $R(t)$ for the times $t = 0, 1.363, 2$.

Doing so gives that $R(0) = 0$, $R(1.363) = 854.5$, and $R(2) = 120$.

Clearly, the maximum occurs at time $t = 1.363$ hours.

Problem — 2015 AP Calculus AB FRQ



Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing?
Give a reason for your answer.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

Solution to part a: We will first identify our critical points. We want $f'(x)$ to be equal to 0. When we scan the graph, clearly that will be true when $x = -2, 1, 4$. We also know that there will be a relative maximum whenever $f'(x)$ changes from positive to negative. This occurs at $x = -2$ which is our relative maximum.

Solution to part b: The function f will be concave down when $f''(x) < 0$ and it will be decreasing when $f'(x) < 0$.

We want $f'(x)$ to be negative (since that means f is decreasing). At the same time, we want $f'(x)$ to be decreasing.

Thus, the intervals are $-2 < x < -1$ and $1 < x < 3$ since that is when $f'(x)$ will be negative (to satisfy the condition of f decreasing) and it is also when $f'(x)$ will be decreasing (to satisfy the concave down condition).

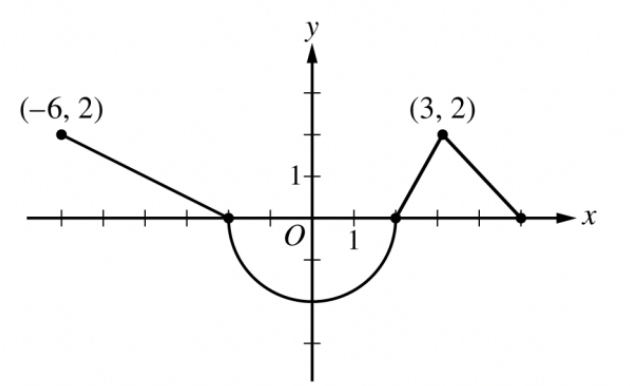
Solution to part c: A point of inflection occurs when the sign of $f''(x)$ changes. To find such points, we can find the points for which $f''(x) = 0$. Then, we can check to the left and to the right of each point to observe sign changes.

By utilizing our graph for $f'(x)$, $f''(x)$ will be 0 at $x = -1, 1$, and 3.

There is indeed a point of inflection at each of those points.
There is a point of inflection $x = -1$ and $x = 3$ because $f'(x)$ changes from decreasing to increasing in value at those points. Note, that if $f'(x)$ is increasing, then $f''(x)$ is positive while if $f'(x)$ is decreasing, then $f''(x)$ is negative.

There is also a point of inflection at $x = 1$ because $f'(x)$ changes from increasing to decreasing in value at that point.

Problem — 2017 AP Calculus AB FRQ



Graph of f'

The function f is differentiable on the closed interval $[6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

I removed parts a and c since it requires content that will be covered in a future unit.

(b) On what intervals is f increasing? Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

Solution to part b: f will be increasing whenever $f'(x)$ is positive.
From the graph, we can tell that $f'(x) > 0$ for the intervals $[-6, -2]$ and $[2, 5]$.

Solution to part d: We can find $f''(-5)$ by finding the slope at $x = -5$ for our f' graph. Clearly, $f''(-5) = \frac{2-0}{-6-(-2)} = -\frac{1}{2}$.

On the other hand, we can't find $f''(3)$ in such a simple manner. The reason is that a sharp turn occurs at $x = 3$ which can be seen in our graph for $f'(x)$.

This is when our limit definition of the derivative comes in.

$$\lim_{x \to 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1 \text{ and } \lim_{x \to 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2$$

The above represents the right-hand limit and the left-hand limit. The limit is also the definition of a derivative in terms of a limit. Since the left-hand limit and right-hand limit differ, $\lim_{x\to 3} \frac{f'(x)-f'(3)}{x-3}$ doesn't exist. This ultimately means that $f''(3)$ doesn't exist.

Unit 6 Integration and Accumulation of Change

§6.1 Exploring Accumulations of Change

Definition 6.0.1

What is an Integral?
An integral is the area under a curve. It's different from a derivative since the derivative was the slope of a curve at a specific point.

We will often be given a **rate** of change function. Sometimes it will be an equation while other times it might be a graph. The area under the graph will represent the **accumulation** of change.

Problem — Pretend someone travels at a speed of 4km/hr for 5 hrs. Find the total distance travelled in km.

Solution: This is an example of a rate of change function. We know that the speed is 4. Since this occurs for 5 hours, the total distance travelled is $4 \cdot 5 = 20$ km.
The problem should give you a basic idea of how integrals and accumulation of change works.

Some problems will give you a basic graph and ask you to find the area under it. Sometimes, the graph can be broken down into simple shapes such as circles, rectangles, and squares. Make sure to break the graph up and sum each part up to find the total area.

Also, be consistent with your signs. When the graph is under the x-axis, you will need to subtract that area. However, if it is above the x-axis, then you add up that part.

§6.2 Approximating Areas with Riemann Sums

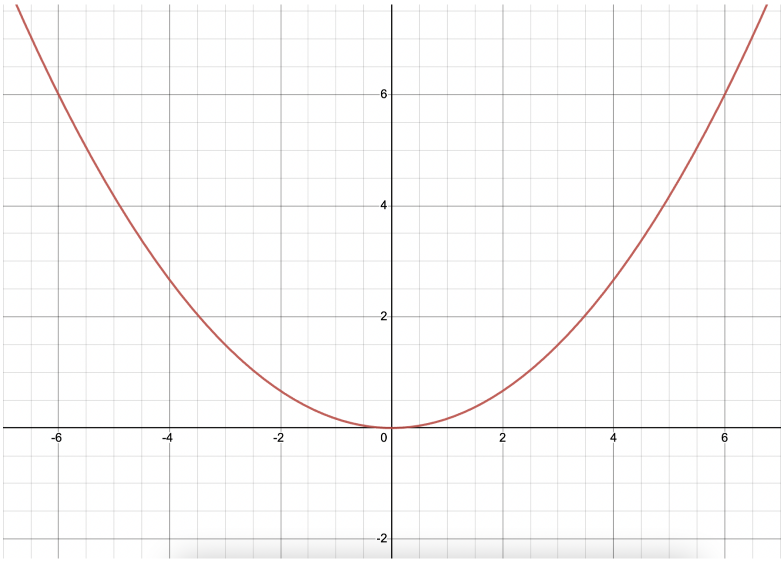
Problem —
We will be given an interval to integrate under. Given the number of rectangles/-subintervals, we can find the width of each rectangle/subinterval.

If the interval is $[a, b]$ and the number of rectangles used for our Riemann sums is n, then the width of each rectangle will be $\frac{b-a}{n}$

Riemann Sums are a useful way to approximate the area under a curve. Riemann sums involve adding up either rectangles of trapezoids to find the area under the curve.

In problems involving Riemann sums, we will be given an interval to find the area under. We will also be given a certain number of subintervals.

This can be best explained with an example so get ready for one.



Find the area under the curve above for the interval $[0, 6]$ and use 3 sub-intervals. Find the area 2 times.

For the first time, use Left-Riemann sums.

For the second time, use Right-Riemann sums.

First, we will find the width of each interval. To do so, we find the difference between our main interval that we want to find the area under. In this case, the length is $6 - 0 = 6$. Now, we will divide the difference by the number of sub-intervals.
Doing so gives $\frac{6}{3} = 2$.

This means we have 3 sub-intervals each of width 2.

Left-Riemann sum means that the left endpoint of the rectangle will lie on the curve. The left endpoint will always be the start of each sub-interval. In this case, the start of each sub-interval will be 0, 2, and 4.

The height of each rectangle will stem from the left endpoint. That means the first rectangle will have a height of $f(0)$. The second one will have a height of $f(2)$. The third one will have a height of $f(4)$.

Since each rectangle has a width of 2, the sum of the areas can be represented as $2[f(0) + f(2) + f(4)]$

Using the curve, we can find the values of $f(0)$, $f(2)$, and $f(4)$ to get $2[0+0.75+2.75] = 7$

For a **Right-Riemann sum**, the right endpoint of each sub-interval will lie on the graph. The right endpoints for each rectangle will occur at $x = 2$, $x = 4$, and $x = 6$.

The height of each rectangle will now stem from the right endpoint. That means the first rectangle will have a height of $f(2)$. The second one will have a height of $f(4)$.

The third will have a height of $f(6)$.

Since each rectangle has a width of 2, the sum of the areas can be represented as $2[f(2) + f(4) + f(6)]$.
Using the curve, we can estimate the values of $f(2)$, $f(4)$, and $f(6)$ to get $2[0.75+2.75+6] = 19$ for the area.

There is also a midpoint Riemann sum. This means that the midpoint of each rectangle will lie on the curve. An example for this will be shown later. You can also encounter a Trapezoidal Riemann sum. To see visualizations of the different types of Riemann sums that you might encounter on the AP exam, make sure to check out the AP Calculus AB/BC Unit 6 Rapid Review video on the TMAS Academy youtube channel.

This might still be confusing. However, don't worry. The next section should help you understand this topic more. There will be more images to help you. There will also be practice problems at the end of the unit.

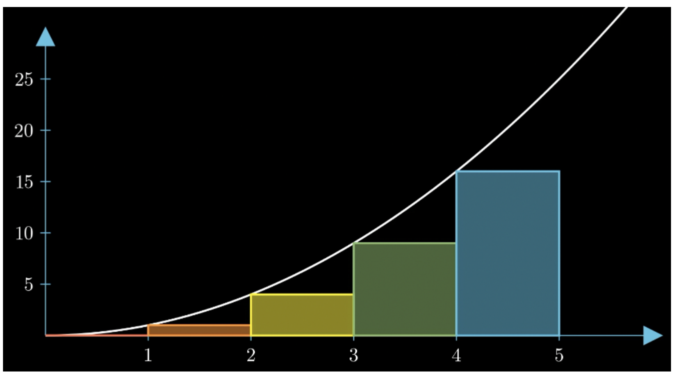
§6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

Before we go deeper into Riemann sums, let's review the types of Riemann Sum methods that we have.
There are 4 types of Riemann sums.
Right Riemann sums is about having the top right vertex (if the curve is above 0 for that point) or bottom right vertex (if curve is below 0) of all rectangles to be touching the graph.

However, in **left Riemann sums**, the top left vertex (if the curve is above 0 for that point) of all rectangles will be touching the graph. The height of a rectangle in left Riemann sums will come from the value of the function at the leftmost point of each base.

On the other hand, the height of a rectangle in right Riemann sums will come from the value of the function at the rightmost point of each base.

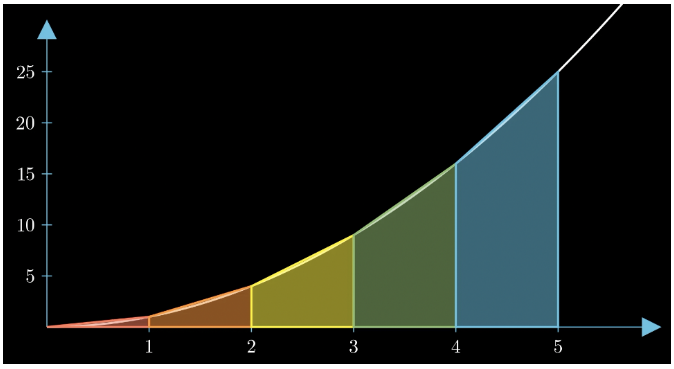
Don't forget to check the AP Calculus AB/BC Unit 6 Rapid Review video on the TMAS Academy youtube channel to see visualizations for all of this!



The above is an example of a left Riemann Sum. The reason is that the top left vertex for each rectangle is on the curve.

For **midpoint Riemann Sums**, we'll simply calculate the height for each rectangle by using the midpoint of each rectangle's base. This means that the height of each rectangle will be the value of the function at the midpoint of the base.

The last one is a trapezoid Riemann Sums. For that, the trapezoid will have bases of lengths that come from the value of the function at the leftmost point of the base and the rightmost point of the base. The image below should help you visualize a trapezoid Riemann Sums.



In general, the number of subintervals will be extremely large. n will be a very large number that approaches infinity.

If the interval we want to find the area under is $[a, b]$, then the width of each subinterval will be $\frac{b-a}{n}$

The height of each rectangle will be $f(a + \frac{b-a}{n}k)$, where k represents the kth subinterval.

Since n approaches infinity, we can represent the area under the curve using a limit.

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\frac{b-a}{n}) \cdot f(a + \frac{b-a}{n}k)$$

The complicated expression above for the area under the curve of $f(x)$ for the interval $[a, b]$ can be represented in a much simpler manner.

That is when our integral sign comes in.
The same area can be represented as $\int_a^b f(x)dx$.

The above is the form of an integral. $f(x)$ represents the function that we are finding the area for. a and b represent the x-coordinates between which we are finding the area (for that specific interval).
means we will find the area for $f(x)$ between $x = a$ and $x = b$. dx represents a small change in x.

Theorem 6.0.1

Definite integral notation
The area under $f(x)$ over the interval $[a, b]$ is

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\frac{b-a}{n})\cdot f(a+\frac{b-a}{n}k)$$

$$\int_a^b f(x)dx$$

Some problems will simply ask us to convert the definite integral to summation notation. We can do that by using the formula above.

Problem — Write out this integral $\int_{-4}^{4} x^3 dx$ in summation notation.

Solution: This can be done by first computing the $\frac{b-a}{n}$ part. The value of it is $\frac{4-(-4)}{n}$ which is $\frac{8}{n}$.

$f(a+\frac{b-a}{n}k)$ simplifies to $f(-4+\frac{8}{n}k)$. Since $f(x) = x^3$, we know that $f(-4+\frac{8}{n}k)$ $= (-4+\frac{8}{n}k)^3$.

Thus, the integral in summation notation is simply

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{8}{n}\cdot(-4+\frac{8}{n}k)^3$$

There will be more examples for this section at the end of this unit, so don't worry if you're still confused.

§6.4 The Fundamental Theorem of Calculus and Accumulation Functions

Theorem 6.0.1

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

In the above, a is a constant while f is a continuous function.

There's another way to write out the fundamental theorem of calculus.

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

$h(x)$ and $g(x)$ in the formula above are differentiable functions that are used as the integration bounds.

The Fundamental Theorem of Calculus makes it easy to take the derivative of an integral.

It's important to know that derivative and integrals are inverses of each other.

Problem — If $F(x) = \int_{2x}^{x^2} \tan(t)dt$, what is $F'(x)$?

Solution: We can apply the fundamental theorem of calculus.
We know that

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

Using the format of the formula above, $h(x)$ is $2x$ while $g(x)$ is x^2.

Plugging this in gives that $F'(x)$ is $\tan(x^2) \cdot 2x - \tan(2x) \cdot 2$.

§6.5 Interpreting the Behavior of Accumulation Functions Involving Area

Accumulation functions are written in this manner: $g(x) = \int_a^x f(t)dt$

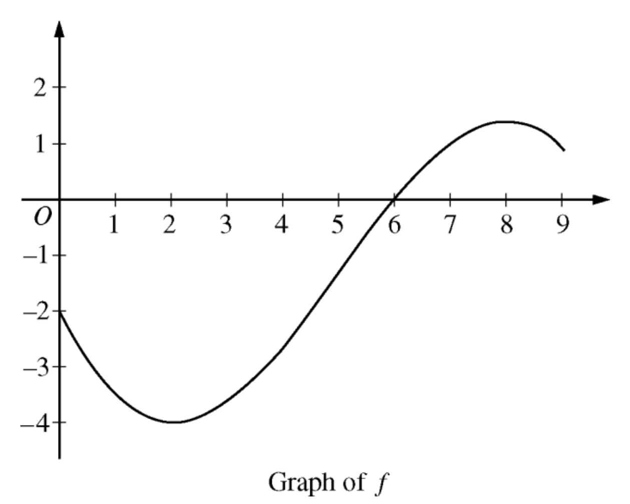
Note that since a is a constant, $g'(x) = f(x)$

In unit 5, we found the relative extrema using the first derivative test.
We can do something similar for accumulation functions.

Since wek now that $g'(x) = f(x)$, for $g(x)$ to be increasing, we can say that $f(x)$ has to be > 0.
On the other hand, for $g(x)$ to be decreasing, we can say that $f(x)$ has to be < 0. The

reason is that $g'(x) = f(x)$ so we can work with $f(x)$ to determine whether $g(x)$ is increasing or decreasing.

Problem — 2012 AP Calculus BC MCQ



The graph of a differentiable function f is shown above. If $h(x) = \int_0^x f(t)dt$, which of the following is true.

(a) $h(6) < h'(6) < h''(6)$
(b) $h(6) < h''(6) < h'(6)$
(c) $h'(6) < h(6) < h''(6)$
(d) $h''(6) < h(6) < h'(6)$
(e) $h''(6) < h'(6) < h(6)$

Solution: $h(6)$ is clearly negative because the curve between $x = 0$ and $x = 6$ is below the x-axis. This means that the area will be negative.

$$h'(x) = \frac{d}{dx}\int_0^x f(t)dt$$

Using the fundamental theorem of calculus, this simplifies to $h'(x) = f(x)$.

This means $h'(6) = f(6)$.
Instead of finding $h'(6)$, we can find $f(6)$ since they both havae the same value. $f(6)$ can be observed in the graph to be 0.

Since $h'(x) = f(x)$, it's obvious that $h''(x) = f'(x)$.

This means $h''(6) = f'(6)$. By looking at the graph, it's obvious that $f'(6)$ is positive since the function f is increasing at that point. This means that $h''(6)$ is also positive. Since $h(6)$ is negative, $h'(6)$ is 0, and $h''(6)$ is positive, we can conclude that

$$h(6) < h'(6) < h''(6)$$

§6.6 Applying Properties of Definite Integrals

Theorem 6.0.1

Definite Integral Properties

$$\int_a^a f(x)dx = 0$$

The above rule is pretty self explanatory. If you're integrating over an interval of length 0, then why would there be an area under the curve (when there is none).

$$\textbf{Reversing Bounds:} \quad \int_a^b f(x)dx = -\int_b^a f(x)dx$$

When you reverse the bounds, then the area will have the same magnitude. The only difference is that the sign will reverse.

$$\textbf{Constants} \quad : \int_a^b kf(x)dx = k \cdot \int_a^b f(x)dx$$

Whenever you have a constant while integrating, you can factor the constant out before integration.

Theorem 6.0.2

More Definite Integral Properties

$$\textbf{Addition:} \quad \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

If you're asked to integrate the sum of two functions, then you can separately integrate each individual function. The reverse of this will also be true as long as both of the integrals have the same integration bounds!

$$\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b [f(x) + g(x)]dx$$

$$\textbf{Subtraction:} \quad \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

If you're asked to integrate the difference between two functions, then you can simply integrate each function separately. The reverse of this will also be true as long as both have integrals have the same bounds.

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

Theorem 6.0.3

Adjacent Intervals
For adjacent intervals in which $a < c < b$, the following rule applies.

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

§6.7 The Fundamental Theorem of Calculus and Definite Integrals

The **antiderivative** of the function $f(x)$ is $F(x)$ such that $F'(x) = f(x)$.

An antiderivative is found by evaluating the indefinite integral. A constant C will always be involved!

Theorem 6.0.1

Antiderivative Power Rule
Let's say that the antiderivative of my function $f(x)$ is $F(x)$.
If $f(x) = x^n$, then

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

Theorem 6.0.2

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)dx = \sin(x) + C$$

Theorem 6.0.3

The Fundamental Theorem of Calculus
If f is a continuous function over the interval $[a, b]$, then the area under the curve of $f(x)$ from that interval can be represented using the antiderivative $F(x)$.

$$\int_a^b f(x)dx = F(b) - F(a)$$

To solve problems involving the fundamental theorem of calculus, just apply the formula above. It's pretty straight-forward.

Problem — 2012 AP Calculus BC MCQ 10

$$\int_1^4 t^{-3/2} dt$$

Solution: We will apply the fundamental theorem of calculus to solve this.
The antiderivative of $t^{-3/2}$ is $-2t^{-1/2}$

$$\text{This means that } \int_1^4 t^{-3/2} dt = -2t^{-1/2}|_1^4$$
$$= -2(4)^{-1/2} - (-2(1)^{-1/2}) = -1 - (-2) = 1$$

§6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

This section connects with section 6.7, so make sure to check that one out if you skipped it.

So far we have seen a few **definite integrals**. We were given bounds to find the area. However, now we will dive into **indefinite integrals**. The process of integrating is completely the same. The only difference is that it is a general solution. This means we must add our constant C when necessary. This might be confusing right now, but some examples will make it clear.

Note
If a term with e doesn't contain any variables, then it is a **constant**. For example, e, e^2, e^3, etc are all constants. The same applies for π.

Problem —

$$\text{Find} \quad \int \left(e^x + e^3\right) dx$$

Solution: The integral of e^x is simply e^x. We can confirm this is true by differentiating e^x and arriving at the same value.
The integral of e^2 is e^2x. The reason is that e^2 is a constant. It's the same thing as integrating a term like 5 which would give us $5x$.

Since this is an indefinite integral, we can't forget our constant. Our answer is $e^x + e^3x + C$.

Note
Make sure to convert all radicals to exponential form. It's extremely easy to integrate something that's in exponential form. For example, $\sqrt{x}$ should be written as $x^{\frac{1}{2}}$.

Problem —

$$\text{Find } \int \frac{x}{x^2 \cdot \sqrt[3]{x}} dx$$

Solution: We will first convert this to exponential form. The denominator is $x^2 \cdot x^{\frac{1}{3}}$. The exponents add to give us $x^{\frac{7}{3}}$.

$$\text{This means we're supposed to evaluate } \int \frac{x}{x^{\frac{7}{3}}}$$

$$\text{We can further simplify the inside part to get } \int x^{-\frac{4}{3}}$$

Using the power rule, we get an answer of $-3x^{-\frac{1}{3}}$.

Note

Often, if you have a polynomial in the numerator and a single variable in the denominator, dividing the polynomial by that variable can be a good strategy before integrating.

Problem —

$$\text{Find } \int (\frac{x^2 + 4x + 3}{x}) dx$$

Solution: As stated in the note above, we'll divide each term by x.

This simplifies $\frac{x^2+4x+3}{x}$ to $x + 4 + \frac{3}{x}$. Now we can integrate this.

$$\int (x + 4 + \frac{3}{x}) dx = \int x dx + \int 4 dx + \int \frac{3}{x} dx = \frac{x^2}{2} + 4x + 3 \ln |x|$$

§6.9 Integrating Using Substitution

Hopefully you remember the Chain Rule from before. Something similar exists for integrals. It's known as **u-substitution** (integration by substitution).

u-substitution allows us to simplify a complicated integral into a simpler one.

We do this by substituting the variable u instead of a complex expression.
For example, if we are asked to find $\int (\sqrt{x+1}) dx$, we can do this through u-substitution.

We can say that $u = x + 1$. Substituting this in gives $\int (\sqrt{u}) dx$

However, now we need to replace dx with du. We do this by differentiating our substitution with respect to x.

Differentiating $u = x + 1$ with respect to x gives $\frac{du}{dx} = 1$. We multiply both sides by dx to get $du = dx$.
We can simply plug in du instead of dx to get $\int(\sqrt{u})du$.
Now we can easily integrate this to get $\frac{2u^{\frac{3}{2}}}{3} + C$
We now need to plug back in $u = x + 1$.
This gives us that the indefinite integral is

$$\frac{2(x+1)^{\frac{3}{2}}}{3} + C$$

A common error is trying u-substitution for the entire denominator while solving inverse trigonometric functions. This is a major error and should be avoided.

Problem —

What is $\int \frac{1}{\sqrt{1-9x^2}} dx$

Solution: Since we have a fraction with a square root expression in the denominator, we should think about inverse trigonometric integration. We should remember our rule that the derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$

We can get a similar expression by making the substitution $u = 3x$.
This turns the integral into

$$\int \frac{1}{\sqrt{1-u^2}} dx$$

However, we can't forget to rewrite dx in terms of du.
This can be done by differentiating $u = 3x$ with respect to x. This gives $du = 3 \cdot dx$.
This means that $dx = \frac{du}{3}$. We can plug this in to get

$$\int \frac{1}{3\sqrt{1-u^2}} du$$

$$\int \frac{1}{3\sqrt{1-u^2}} du = \frac{\sin^{-1}(u)}{3} + C$$

Now, we plug in our expression for u which was $u = 3x$ to get our final answer.

$$\frac{\sin^{-1}(3x)}{3} + C$$

Note

Often, when you are supposed to integrate something consisting of two expressions multiplied/divided to each other with one having a higher degree, then u-substitution is a key strategy.

This works when the expression with the higher degree has a degree that is exactly 1 higher.
This might be confusing, but the example below should clarify it.

Problem — Use u-substitution to rewrite this integral! No need to solve for a numerical answer.

$$\int_0^4 (3x^3\sqrt{x^4+1})dx$$

Solution: We can make the substitution $u = x^4 + 1$.
This turns the expression into $\int_0^4 (3x^3\sqrt{u})dx$

Now, we can't forget to rewrite dx in terms of du. This can be done by differentiating $u = x^4 + 1$ with respect to x.
That gives $du = 3x^3 \cdot dx$

Using the $3x^3$ before $\sqrt{u}$ in our expression and dx, we can rewrite our integral with du.

$$\int_0^4 \sqrt{u}du$$

However, we can't forget that our bounds are wrong. Our bonds from 0 to 4 were the case when we were integrating with respect to x. To find the new bounds, we plug in 0 and 4 into the substitution ($u = x^4 + 1$). This gives us 1 and 257. These are our new bounds when our integral is in terms of u.

$$\int_1^{257} \frac{2u^{\frac{3}{2}}}{3}$$

This concludes the problem. We found a different expression for the integral using u-substitution.

§6.10 Integration with Long Division and Completing the Square

Integration with long division works really well when the numerator has a higher degree. You simply divide the denominator from it and integrate the new expression.

Problem —

$$\int (\frac{x^3+1}{x+2})dx$$

Solution: Through long division, we can simplify $\frac{x^3+1}{x+2}$ to $x^2 - 2x + 4 - \frac{7}{x+2}$

Now, we simply evaluate $\int (x^2 - 2x + 4 - \frac{7}{x+2})dx$

$$\int (x^2 - 2x + 4 - \frac{7}{x+2})dx = \int x^2dx + \int -2xdx + \int 4dx + \int -\frac{7}{x+2}dx$$

Using the anti-derivative of the power rule and the fact that the derivative of $\ln(x)$ is $\frac{1}{x}$, we can quickly find the integral to be $\frac{x^3}{3} - x^2 + 4x - 7\ln|x+2|$

Integrating with completing the square works well, especially when the denominator of the term that you want to integrate contains a quadratic.

If the x^2 term has a negative coefficient, then you factor out the negative sign before doing anything.

For example, if I am supposed to complete the square for $-x^2 - 6x - 5$, we will first out the negative sign.

$$-x^2 - 6x - 5 = -(x^2 + 6x + 5)$$

Now we will complete the square to get $-(x+3)^2 + 4$

Problem —

$$\int \frac{1}{x^2 + 16x + 65} dx$$

Solution: We will first complete the square in the denominator.
We notice that $x^2 + 16x + 65 = (x^2 + 16x + 64) + 1 = (x+8)^2 + 1$

We can use the expression $(x+8)^2 + 1$ instead of $x^2 + 16x + 65$

Our integral becomes $\int \frac{1}{(x+8)^2+1} dx$

We should recognize that this look similar to $\frac{1}{x^2+1}$ which is the derivative of $\tan^{-1}(x)$.
We want to try to convert it to a similar expression.
We can do that by substituting $u = x + 8$.

$$\int \frac{1}{u^2 + 1} dx$$

However, we can't forget to rewrite our dx in terms of du. This can be done by differentiating our substitution $(u = x + 8)$ with respect to x. That gives $du = dx$.

Substituting that simplifies the integral to $\int \frac{1}{u^2+1} du$ which is $\tan^{-1}(u) + C$

Now, we plug in our substitution back to get an answer of $\tan^{-1}(x + 8) + C$

Often, as seen in the example above, integration through completing the square utilizes an inverse trigonometric function.

§6.14 Selecting Techniques for Antidifferentiation

This section doesn't have any new content. Just review all our antidifferentiation techniques. You need to know when to use a specific technique. You will be applying your techniques in our cumulative practice for unit 6.

Unit 6 Practice Problems

Problem — 2009 AP Calculus AB FRQ

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

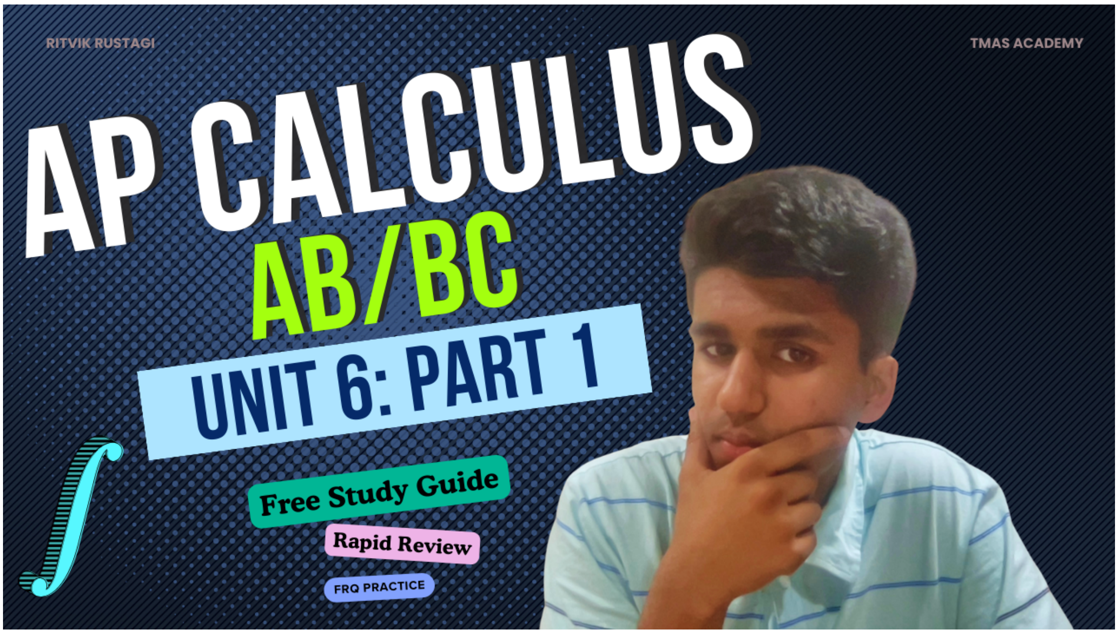
Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \leq x \leq 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

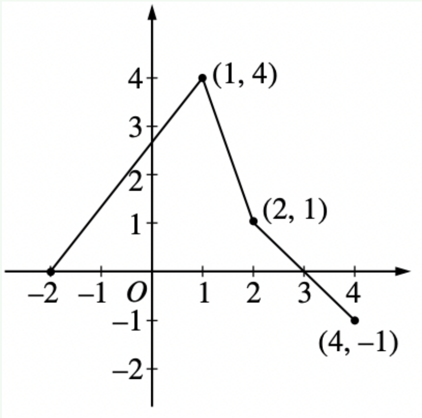
(b) Evaluate $\int_2^{13}(3 - 5f'(x))dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13}(3 - 5f'(x))dx$. Show the work that leads to your answer.

Solution: Video Solution



Problem — 1999 AP Calculus BC FRQ



The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

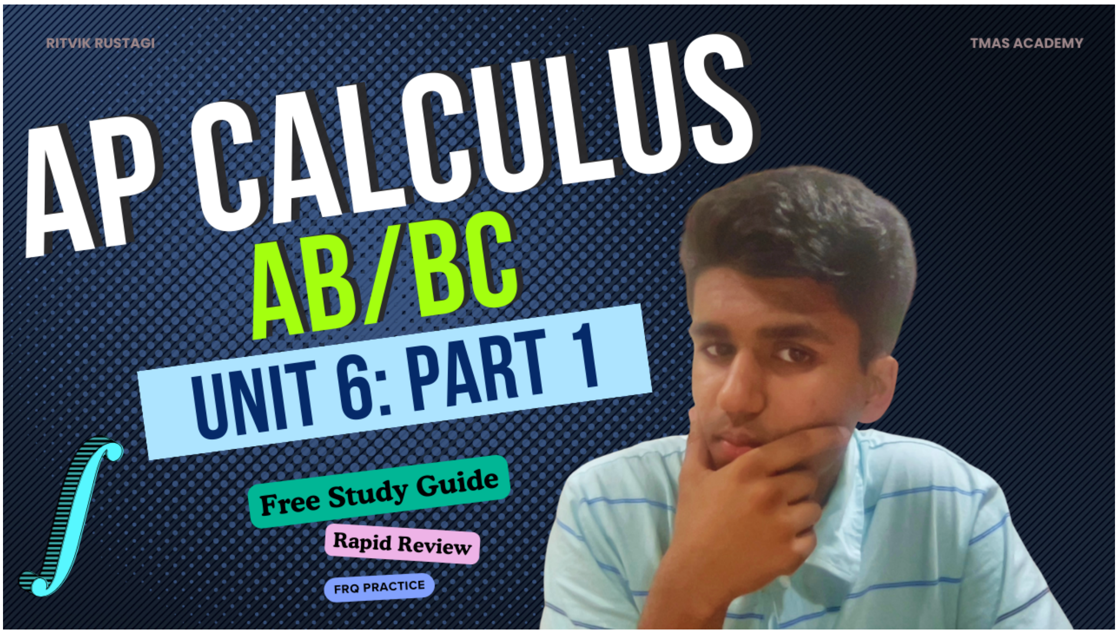
(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

Solution: Video Solution



Problem — 2001 AP Calculus AB FRQ

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \leq t \leq 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days

Solution to part a: We will choose 1 time that is closest to 12 from the left and 1 time that is closest to 12 from the right to approximate $W'(12)$.

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = \frac{21 - 24}{6} = -0.5°C/\text{day}$$

Solution to part b: The average temperature is the sum of all temperatures divided by the interval of time which is 15.

We can approximate the sum of all temperature by finding the integral using our trapezoidal Riemann sum method.
Since our subinterval has length 3, we will have $\frac{15}{3} = 5$ trapezoids to work with.
The area of the first trapezoid will be $\frac{W(3)+W(0)}{2} \cdot 3$.

The area of the second trapezoid will be $\frac{W(6)+W(3)}{2} \cdot 3$

The area of the third trapezoid will be $\frac{W(9)+W(6)}{2} \cdot 3$

The area of the fourth trapezoid will be $\frac{W(12)+W(9)}{2} \cdot 3$

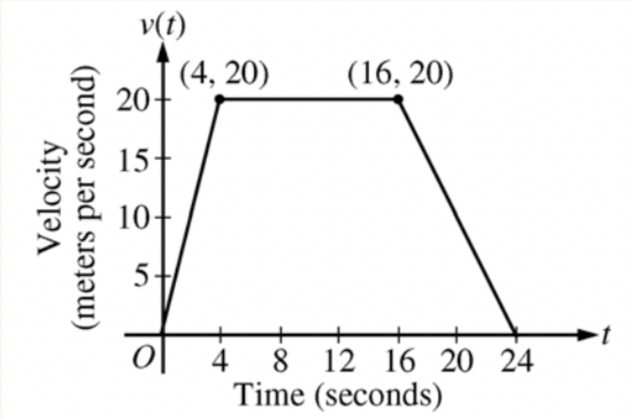
The area of the fifth trapezoid will be $\frac{W(15)+W(12)}{2} \cdot 3$

We can add all of the areas up to get

$$\frac{3}{2}(W(0) + 2W(3) + 2W(6) + 2W(9) + 2W(12) + W(15)) = 376.5C$$

Now we can divide this by the interval length which is 15. Doing so gives that the average temperature is $\frac{376.5}{15} = 25.1°C$

Problem — 2005 AP Calculus AB FRQ



A car is traveling on a straight road. For $0 \leq t \leq 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

Find $\int_0^{24} v(t)dt$. Using the correct units, explain the meaning of $\int_0^{24} v(t)dt$.

Solution: This particular problem was included since it will allow you to understand how sometimes an integral can be found by finding the area under the curve by using the common shapes that we constantly use. For example, in this problem, the area under the curve can be found by splitting up our curve into a few triangles and rectangles.

The area is $\frac{4 \cdot 20}{2} + 12 \cdot 20 + \frac{8 \cdot 20}{2} = 360$ meters.
This means that the car travels 360 meters in 24 seconds.

Problem — 2006 AP Calculus AB FRQ

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

Solution to part a: We know that the average acceleration is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80 - 0} = \frac{11}{20}\text{ft/sec}^2$$

For those that forgot, the average rate of change is how a function changes on average for an entire interval. For example, the average rate of change for a function $f(x)$ for the interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$

Solution to part b: To use midpoint Riemann sums to approximate $\int_{10}^{70} v(t)dt$, we first find the width of each rectangle.

Our interval width is $70 - 10 = 60$. We divide this by the number of subintervals to find that each subinterval is $\frac{60}{3} = 20$ units wide.

For the first subinterval from $t = 10$ to $t = 30$, the rectangle will have an area of $20 \cdot v(\frac{10+30}{2}) = 20 \cdot v(20)$

The second subinterval will have an area of $20 \cdot v(\frac{30+50}{2}) = 20 \cdot v(40)$

The third subinterval will have an area of $20 \cdot v(\frac{50+70}{2}) = 20 \cdot v(60)$

We can sum up all the areas to find that the total area using midpoint Riemann sums is $20[v(20) + v(40) + v(60)]$.

We can use our table to plug in our known values for $v(20)$, $v(40)$, and $v(60)$. Doing so gives that the area is $20(22 + 35 + 44) = 2020$ ft.

$\int_{10}^{70} v(t)dt$ represents the rocket's displacement from time $t = 10$ to time $t = 70$.

Solution to part c: We will integrate $a(t)$ to find $v(t)$ for rocket B.

$$v(t) = \int a(t)dt = \int \frac{3}{\sqrt{t+1}}dt$$

This is a problem that involves u-substitution. Plugging in $u = t + 1$ makes our integral a lot simpler. The reason is that we can differentiate both sides of our substitution to get $du = dt$.

Plugging in both gives that $v(u) = \int \frac{3}{\sqrt{u}}du$

Now, we can integrate this to find that $v(u) = 6\sqrt{u} + C$
We can substitute $u = t + 1$ back again to find that $v(t) = 6\sqrt{t+1} + C$

Now, we must solve for our constant C.

We know that the initial velocity (at time $t = 0$) is 2. We can plug $t = 0$ and $v = 2$ to get $2 = 6\sqrt{0+1} + C = 6 + C$
Solving for C gives that $C = -4$.

This means that expression for velocity is $v_B(t) = 6\sqrt{t+1} - 4$ (a subscript of B was added to indicate that this expression for velocity is specifically for rocket B)

We can plug in 80 to find that $v_B(80) = 6\sqrt{80+1} - 4 = 6 \cdot 9 - 4 = 50$ ft/s

From our given table, we know that at time $t = 80$, $v_A(80) = 49$ ft/s. Clearly, v_B is larger which means that rocket B is travelling faster at time $t = 80$ seconds.

Problem — 2022 AP Calculus AB FRQ
Particle P moves along the x-axis such that, for time $t > 0$, its position is given by $x_P(t) = 6 - 4e^{-t}$. Particle Q moves along the y-axis such that, for time $t > 0$, its velocity is given by $v_Q(t) = \frac{1}{t^2}$. At time $t = 1$, the position of particle Q is $y_Q(1) = 2$.

(a) Find $v_P(t)$, the velocity of particle P at time t.

(b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

(c) Find $y_Q(t)$, the position of particle Q at time t.

Solution to part a: We have an expression for the position given. We can simply differentiate it to find an expression for velocity.

$$v_P(t) = \frac{d}{dt}[x_P(t)] = \frac{d}{dt}[6 - 4e^{-t}] = 4e^{-t}$$

Solution to part b: We know that the derivative of velocity with respect to time gives acceleration.

$$a_Q(t) = \frac{d}{dt}[v_Q(t)] = \frac{d}{dt}[\frac{1}{t^2}] = -\frac{2}{t^3}$$

The speed of the particle Q will be decreasing when the signs of velocity and acceleration will be different. On the other hand, the speed of any object in general will be increasing when velocity and acceleration have the same sign.

For any time $t > 0$, $v_Q(t) = \frac{1}{t^2}$ will always be positive. Similarly, the acceleration $a_Q(t) = -\frac{2}{t^3}$ will always be negative. Thus, since $v_Q(t) > 0$ and $a_Q(t) < 0$, the speed of particle Q will be decreasing for all times.

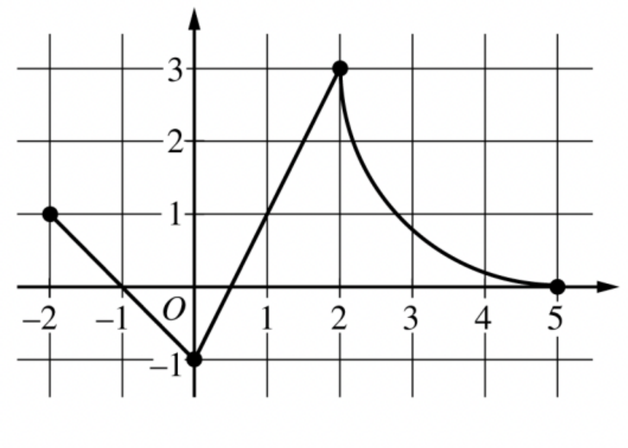
Solution to part c: To find an expression for the position for particle Q, we must first integrate its expression for velocity.

$$x_Q(t) = \int v_Q(t)dt = \int \frac{1}{t^2}dt = -\frac{1}{t} + C$$

Since we know that the position of particle Q at time $t = 1$ is 2, we can plug in that point to get $2 = -\frac{1}{1} + C = -1 + C$
We can rearrange to find that $C = 3$.

We can plug in our constant back to find that $x_Q(t) = -\frac{1}{t} + 3$

Problem — 2019 AP Calculus AB FRQ



Graph of f

The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)dx = 7$, find the value of $\int_{-6}^{-2} f(x)dx$. Show the work that leads to your answer.
(b) Evaluate

$$\int_{3}^{5} (2f'(x) + 4)dx$$

(c) The function g is defined by $g(x) = \int_{-2}^{x} f(t)dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.
(d) Find

$$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$$

Solution to part a: $\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{5} f(x)dx$ because we have adjacent intervals.

To find $\int_{-6}^{-2} f(x)dx$, we just need to find $\int_{-2}^{5} f(x)dx$ and subtract that from $\int_{-6}^{5} f(x)dx$ (which is 7).

To find $\int_{-2}^{5} f(x)dx$, we simply need to find the area under the curve of $f(x)$ for the interval from -2 to 5.

From -2 to -1, our area is $\frac{1}{2}$. From -1 to $\frac{1}{2}$, our area is $-\frac{3}{4}$.

From the interval $\frac{1}{2}$ to 2, our area is simply a triangle with base $\frac{3}{2}$ and height 3. This means the area is $\frac{9}{4}$.

For the interval from 2 to 5, the area under the curve involves subtracting a quarter circle with radius 3 from a square with a side length of 3. The square has area 9. The quarter circle has area $\frac{9\pi}{4}$. This means that the area for this interval is $9 - \frac{9\pi}{4}$.

Adding up the areas for these small intervals gives $\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + 9 - \frac{9\pi}{4}$. This evaluates to $11 - \frac{9\pi}{4}$.

This means that $\int_{-2}^{5} f(x)dx = 11 - \frac{9\pi}{4}$.

We subtract this from 7 to get $\frac{9\pi}{4} - 4$ which is $\int_{2}^{6} f(x)dx$.

Solution to part b:

$$\int_3^5 (2f'(x) + 4)dx = \int_3^5 2f'(x)dx + \int_3^5 4dx = 2\int_3^5 f'(x)dx + \int_3^5 4dx$$

We can use the Fundamental Theorem of Calculus to find the answer.

$$2\int_3^5 f'(x)dx = 2(f(5) - f(3)) = 2(0 - (3 - \sqrt{5})) = -6 + 2\sqrt{5}$$

$$\int_3^5 4dx = 4x|_3^5 = 4(5 - 3) = 8$$

The answer is the sum of the 2 integrals which is $2 + 2\sqrt{5}$.

Solution to part c: To find the absolute minimum of $g(x)$, we will first find the critical points of $g(x)$. Then, we'll use those points and the endpoints of the interval to see the minimum value for $g(x)$.

The critical points of $g(x)$ can be found by differentiating $g(x)$ and seeing when $g'(x)$ equals to 0 or is undefined.

$$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)dt = f(x)$$

Since $g'(x) = f(x)$, we simply need to check when $f(x)$ intersects the x-axis because that's when $f(x)$ is 0 (meaning that $g'(x)$ is also 0).

$g'(x) = f(x) = 0$ at points $x = -1, \frac{1}{2}$, and 5.

The endpoint not included in our critical points is -2.

Thus, we need to find $g(x)$ for $-2, -1, \frac{1}{2}$, and 5.

$$g(-2) = \int_{-2}^{-2} f(x)dx = 0$$

$$g(-1) = \int_{-2}^{-1} f(x)dx = \frac{1}{2}$$

$$g(\frac{1}{2}) = \int_{-2}^{\frac{1}{2}} f(x)dx = -\frac{1}{4}$$

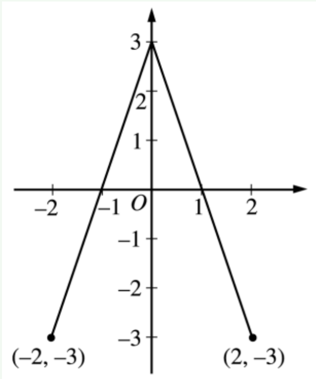
$$g(5) = \int_{-2}^{5} f(x)dx = 11 - \frac{9\pi}{4}$$

Clearly, the absolute maximum value for the interval $-2 \le x \le 5$ is $g(5)$ which is $11 - \frac{9\pi}{4}$.

Solution to part d: We first try direct substitution for our limit to see that it works!

$$\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan(1)} = \frac{10-6}{1-\frac{\pi}{4}} = \frac{4}{1-\frac{\pi}{4}} = \frac{16}{4-\pi}$$

Problem — 2002 AP Calculus AB FRQ



Graph of f

The graph of the function f shown above consists of two line segments. Let g by the function $g(x) = \int_0^x f(t)dt$.

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2, 2)$ is the graph of g concave down. Explain your reasoning.

Solution to part a: We know that $g(-1) = \int_0^{-1} f(t)dt$.
Using the rule that states $\int_a^b f(x)dx = -\int_b^a f(x)dx$, we know that

$$\int_0^{-1} f(t)dt = -\int_{-1}^{0} f(t)dt$$

This means that $g(-1) = -\int_{-1}^{0} f(t)dt$.
We can use our graph to find that the area for $\int_{-1}^{0} f(t)dt$ will simply be a right triangle with a height of 3 and base 1. This means that the area is $\frac{3}{2}$.
Since $\int_{-1}^{0} f(t)dt = \frac{3}{2}$, $g(-1) = -\frac{3}{2}$
We can use the fundamental theorem of calculus to find that

$$g'(x) = \frac{d}{dx}[\int_0^x f(t)dt] = f(x)$$

Plugging in -1 gives that $g'(-1) = f(-1) = 0$

Now we can find $g''(x)$ since we know that $g'(x) = f(x)$. We can differentiate $g'(x)$ once more to find that $g''(x) = f'(x)$.
Plugging in -1 gives $g''(-1) = f'(-1) = 3$

Solution to part b: The function g will be increasing whenever $g'(x)$ is positive. We know that in general, a function is increasing when the first derivative is positive.

We can use the fundamental theorem of calculus to find that

$$g'(x) = \frac{d}{dx}[\int_0^x f(t)dt] = f(x)$$

This means that $g'(x) = f(x)$ needs to be positive for $g(x)$ to be increasing. $f(x)$ will be positive, meaning that it lies above the x-axis, whenever x satisfies this condition: $-1 < x < 1$

Thus, the function g will be increasing for $-1 < x < 1$ because that's when $g'(x) = f(x) > 0$.

Solution to part c: The graph of g will be concave down whenever $g''(x) < 0$. We already found in part a that $g''(x) = f'(x)$.
Thus, $g''(x) = f'(x)$ needs to be < 0.

Rather than finding when $g''(x) < 0$, we can find the values for which $f'(x) < 0$ since $g''(x)$ and $f'(x)$ are the same. We know that at the points where $f'(x) < 0$, $f(x)$ will be decreasing decreasing. From the graph, it's clear that $f(x)$ decreases for $0 < x < 2$.

The graph of g is concave down for $0 < x < 2$ because that's when $g'(x) = f(x)$ is less than 0.

Problem — 2002 AP Calculus AB FRQ

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \leq x \leq 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \leq x \leq 1.5$.

(a) Evaluate $\int_0^{1.5}(3f'(x) + 4)dx$. Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

(c) Find a positive real number r having the property that there must exist a value c with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

Solution to part a: We can use our integral rules to get

$$\int_0^{1.5}(3f'(x) + 4)dx = 3\int_0^{1.5} f'(x)dx + \int_0^{1.5} 4dx$$

$$= 3[f(x)|_0^{1.5}] + 4x|_0^{1.5} = 3[f(1.5) - f(0)] + 4(1.5 - 0)$$

$$= 3(-1 - (-7)) + 4 \cdot 1.5 = 18 + 6 = 24$$

Solution to part b: At $x = 1$, the equation of the tangent line can be found by first finding the slope at that point. We know that $f'(1) = 5$.

We can plug this in for m into $y = mx + b$ which is the equation of our tangent line (and any line in general).

Plugging it in gives $y = 5x + b$. We know that the tangent line intersects $(1, f(1))$ which is $(1, -4)$.

We can plug that point into our equation for the line to get $-4 = 5 \cdot 1 + b$
Solving gives that $b = -9$.

This means that the equation of our tangent line is $y = 5x - 9$

We can plug in $x = 1.2$ to get $y \approx 5 \cdot 1.2 - 9 = 6 - 9 = -3$ (our approximation for $f(1.2)$

We know that the approximation for $f(1.2)$ will be less than the real value if the function f is c down between 1 and 1.2.

Since $f'(x)$ and $f(x)$ both increase from 1 to 1.5, we can tell that the function f is indeed concave up for our interval.

This means that our approximation for $f(1.2)$ is less than the actual value.

Solution to part c: We should recognize this problem is related to the Mean Value Theorem. The problem statement partially states the Mean Value Theorem since it asks us to confirm if a certain rate of change exists.

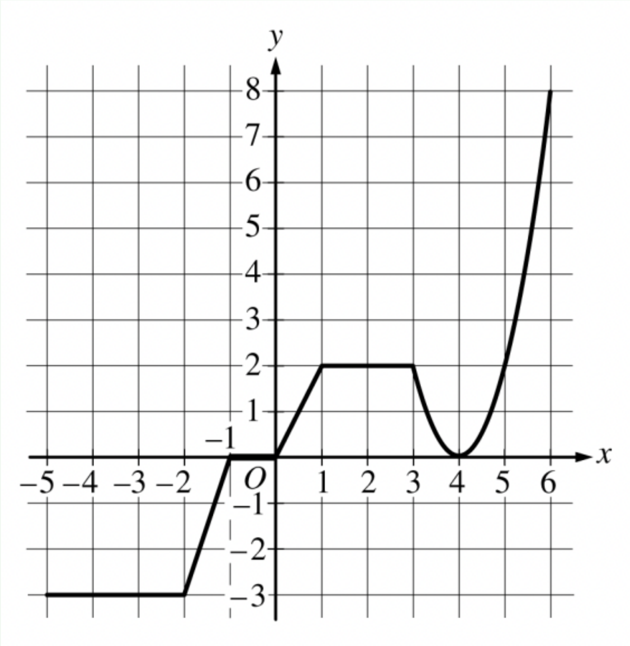
The Mean Value Theorem conditions are satisfied since the problem says that the function f is differentiable for all real numbers.

We know from the Mean Value Theorem that a value c such that $0 < c < 0.5$ will exist to satisfy

$$f''(c) = \frac{f'(0.5) - f'(0)}{0.5 - 0} = \frac{3 - 0}{0.5 - 0} = 6$$

This means that the r in the problem statement is 6 from MVT (Mean Value Theorem).

Problem — 2018 AP Calculus AB FRQ



Graph of g

The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_1^6 g(x)dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Solution to part a: The problem statement states that the graph of $g(x)$ and $f'(x)$ are the same.
We know that $f(-5) = f(1) + \int_1^{-5} f'(x)dx$
We can rewrite this as $f(-5) = f(1) - \int_{-5}^{1} f'(x)dx$
It's important to rewrite the integral since traditionally we are used to finding the area from a point to the left to a point to the right.

Now, instead of writing our expression for $f(-5)$ in term of $f'(x)$, we can just use $g(x)$ since we know that both functions are equal to each other.

$$f(-5) = f(1) - \int_{-5}^{1} f'(x)dx = f(1) - \int_{-5}^{1} g(x)dx$$

We can find $\int_{-5}^{1} g(x)dx$ by computing the area under the graph by dividing it into rectangles and triangles to get $-\frac{19}{2}$.

Plugging this in along with $f(1) = 3$ gives $f(-5) = 3 - (-\frac{19}{2}) = \frac{25}{2}$.

Solution to part b: $\int_1^6 g(x)dx$ can be found by finding the area through some geometry for the interval from 1 to 3. However, from 3 to 6, we can use some of our integration techniques to find the area because our equation is given for that part of the curve.

$$\int_1^6 g(x)dx = \int_1^3 g(x)dx + \int_3^6 g(x)dx$$

$\int_1^3 g(x)dx$ is simply a 2 by 2 rectangle meaning it has an area of 4.

$\int_3^6 g(x)dx$ can be found by integrating $g(x) = 2(x-4)^2$

$\int_3^6 2(x-4)^2dx$ can be found by u-substitution.
We can make the substitution $u = x - 4$ which means $du = dx$.

$$\text{Substituting it in gives } \int_3^6 2u^2du$$

However, there is one thing we forgot to do! We must also change the integration bounds. 3 to 6 was the integration bounds when we were integrating with respect to x.

The new integration bounds can be found by substituting $x = 3$ and $x = 6$ into $u = x - 4$. Doing so gives that the new bounds are $u = -1$ and $u = 2$. We can plug this in to get

$$\int_{-1}^2 2u^2du = \frac{2u^3}{3}|_{-1}^2$$

$$= \frac{2(2)^3}{3} - \frac{2(-1)^3}{3} = 6$$

Since we now have our value of $\int_3^6 g(x)dx$ through the u-substitution method, we can substitute this into $\int_1^6 g(x)dx = \int_1^3 g(x)dx + \int_3^6 g(x)dx$. We also already know that $\int_1^3 g(x)dx$ is simply 4.

The sum of both is 10 which is our answer.

Solution to part c: The graph of $f(x)$ will be both increasing and concave up when $f'(x) > 0$ and $f''(x) > 0$.

Since $f'(x) = g(x)$, to find the intervals for when $f'(x)$ is increasing, we just need to check to see whether or not $g(x)$ is increasing.

$f'(x)$ is increasing whenever $f'(x) = g(x)$ is positive (above the x-axis).
Between the interval from $-5 < x < 6$, $g(x)$ is greater than 0 from 0 to 6 ($0 < x < 6$).

Since $f'(x) = g(x)$, we know that $f''(x) = g'(x)$. Thus, instead of checking for concavity through $f''(x)$, we can use $g'(x)$ since both functions are equal. Our function $f(x)$ will be concave up whenever $g'(x) > 0$.

This is the same thing as saying that $f(x)$ will be concave up whenever $g(x)$ is increasing. $g(x)$ is increasing from the intervals $-2 < x < -1$, $0 < x < 1$, and $4 < x < 6$.

Combining these bounds with the bounds we found from texting $f'(x)$, we see that our answer is $0 < x < 1$ and $4 < x < 6$.

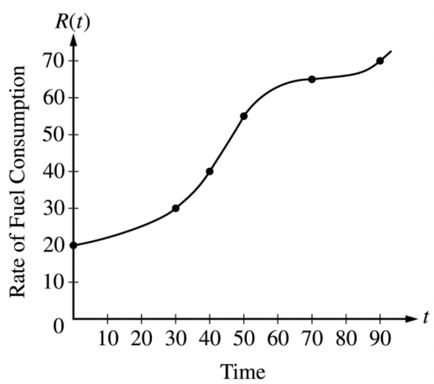
In summary, the intervals for which f is both increasing and concave up is $0 < x < 1$ and $4 < x < 6$ because that's when $f'(x) = g(x) > 0$ and $f''(x) = g'(x) > 0$.

Solution to part d: We know that a point of inflection will occur for $f(x)$ when the sign of $f''(x)$ changes.

We also know that $f''(x) = g'(x)$. This means we can test the signs for $g'(x)$ to see when it changes.

Observing the graph of $g(x)$ gives that the sign of $g'(x)$ only changes at $x = 4$. Thus, $x = 4$ is our inflection point for $f(x)$ because the sign of $f''(x) = g'(x)$ changes since the function g decreases until 4 but increases right after.

Problem — 2003 AP Calculus AB FRQ



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \leq t \leq 90$ minutes, are shown above.

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)dt$? Explain your reasoning.

Solution to part a: We choose the closest value from the table to the left from $t = 45$ and the closet value to the right from $t = 45$.

This means $R'(45) \approx \frac{R(50)-R(40)}{50-40} = \frac{55-40}{50-40} = \frac{3}{2}$ gal/min^2

Solution to part b: Since $R'(t)$ has a maximum value at $t = 45$, we know that $R''(t) = 0$. This is an example of working backwards from our relative extrema. We know that a relative extrema at $x = c$ for $f(x)$ can exist if $f'(c) = 0$ or is undefined.

Solution to part c: We know that in left Riemann sum, the left endpoint of each subinterval will lie on the graph. Also, since we integrate from 0 to 90, our total width will be 90. Since we want 5-subintervals, each subinterval will be $\frac{90}{5} = 18$ wide. However, in this situation, we won't use this method. The reason is that our table already contains specific $R(t)$ values for specific times. We must use those.

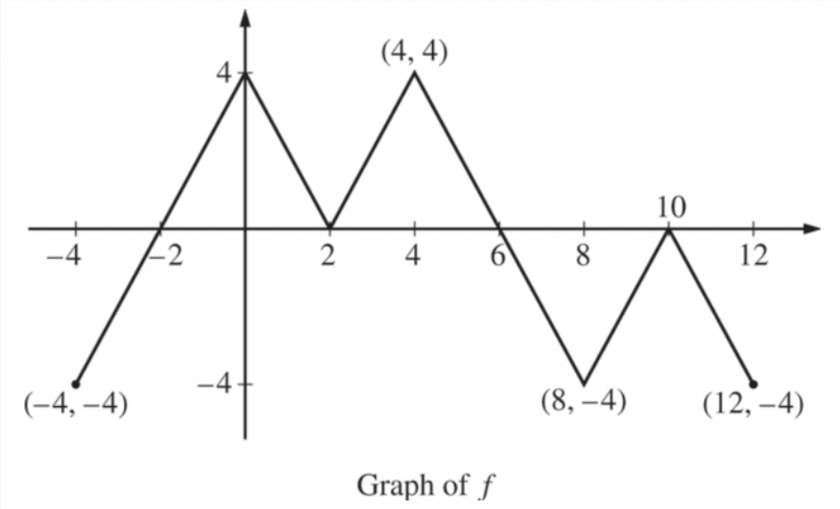
The first rectangle will have width from $t = 0$ to $t = 30$. Similarly, the second rectangle will have width from $t = 30$ to $t = 40$. The third rectangle will have its width from $t = 40$ to $t = 50$ and so on.
The height of each rectangle will stem from the left endpoint. For example, the first rectangle's height will be $R(0)$. Similarly, the second rectangle's height will be $R(30)$.

Using this tells us that the left Riemann sum is

$$30 \cdot R(0) + 10 \cdot R(30) + 10 \cdot R(40) + 20 \cdot R(50) + 20 \cdot R(70) = 30 \cdot 20 + 10 \cdot 30 + 10 \cdot 40 + 20 \cdot 55 + 20 \cdot 65 = 3700$$

Since the graph is increasing in value, our approximation will be less than the true value of $\int_0^{90} R(t)dt$. In general, remember that if the graph is increasing, then the Left Riemann sum will give an underestimate while the Right Riemann sum will give an overestimate.

Problem — 2016 AP Calculus AB FRQ



The figure above shows the graph of the piecewise-linear function f. For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t)dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.

Solution to part a: To find whether or not g has a relative minima/maxima or neither at $x = 10$, we need to differentiate $g(x)$ first.

Since $g(x) = \int_2^x f(t)dt$, we can use the Fundamental Theorem of Calculus to find that $g'(x) = f(x)$.

We know that a relative minimum exists when the first derivative switches signs from negative to positive while a relative maximum exists when the first derivative switches signs from positive to negative.

Instead of investigating with $g'(x)$, we can work with $f(x)$ since both are equal functions. To the left of $x = 10$, $f(x)$ is negative which means $g'(x)$ is also negative; $g(x)$ is decreasing. Similarly, to the right of $x = 10$, $f(x)$ is negative which means that $g'(x)$ is also negative; $g(x)$ is decreasing.

Since $g(x)$ is decreasing to the left and right of $x = 10$, we have neither a relative minimum nor a relative maximum at that point.

Solution to part b: To test whether or not an inflection point exists, we first find the second derivative of $g(x)$ which is represented as $g''(x)$.

We already know from the fundamental theorem of calculus that $g'(x) = f(x)$
We can take the derivative of both sides again with respect to x to get $g''(x) = f'(x)$.

An inflection point will exist if $g''(x)$ switches signs to the left and to the right of that point. Instead of testing the sign change for $g''(x)$, we can test it for $f'(x)$ since both functions are the same.

Clearly, to the left of $x = 4$, $f'(x)$ is positive because $f(x)$ is increasing. However, to the right of $x = 4$, $f'(x)$ is negative because $f(x)$ is decreasing.

Since the sign of $g''(x) = f'(x)$ changes at $x = 4$, an inflection point indeed exists for $g(x)$ at $x = 4$.

Solution to part c: We can find the absolute minima and maxima by finding the critical points first.
The critical points can be found by taking the first derivative of $g(x)$.
We know that $g'(x) = f(x)$ from the fundamental theorem of calculus.

A critical point will exist whenever $g'(x)$ is either 0 or undefined. $g'(x) = f(x)$ has critical points at $x = -2, 2, 6$, and 10.

However, since we want to find the absolute extrema, we only have to work with the critical points that involve a sign change. In general, when you're finding the absolute extrema, you can save time by only working with the critical points with a sign change. If no sign change occurs, then you don't need to test that value since we already know that neither a relative minimum nor a relative maximum will exist at that point.

A sign change only occurs at $x = -2$ and 6.

On top of those 2 points, we must test the endpoints of the interval which are -4 and 12.

Thus, to find the absolute extrema, we simply need to find $g(x)$ for $-4, -2, 6$, and 12. We simply use our area formulas (such as for a triangle) to find the area for each.

$g(-4) = -4$
$g(-2) = -8$
$g(6) = 8$
$g(12) = -4$

Clearly, the absolute maximum occurs at point $x = 6$ for which $g(6) = 8$. The absolute minimum occurs at point $x = -2$ for which $g(-2) = -8$.

Problem — 2015 AP Calculus AB FRQ

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$ Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

Solution to part a: We can estimate $v'(16)$ by finding the slope between $t = 12$ and $t = 20$ since they are the points closest to $t = 16$.

$$v'(16) \approx \frac{v(20) - v(12)}{20 - 12} = \frac{240 - 200}{8} = 5$$

Solution to part b: $\int_0^{40} |v(t)|dt$ represents the distance travelled. The reason is that we take the absolute value of $v(t)$. This means that direction doesn't matter since the absolute value of velocity is always speed. Thus, the answer isn't displacement. It is **distance**.

Using a right Riemann sum with 4 sub-intervals gives that:

$$\int_0^{40} |v(t)|dt \approx |v(12)| \cdot (12-0) + |v(20)| \cdot (20-12) + |v(24)| \cdot (24-20) + |v(40)| \cdot (40-24)$$

The right Riemann sum approximation evaluates to $200 \cdot 12 + 240 \cdot 8 + 220 \cdot 4 + 150 \cdot 16 \approx 7600$ meters

Solution to part c: Since we have the equation for the velocity $B(t)$, we simply differentiate it with respect to t to find $a(t)$.
$a(t) = B'(t) = 3t^2 - 12t$

Now we can plug in $t = 5$ to find that the acceleration is 15 meter/min^2

Solution to part d: To find the average velocity, we first find the displacement.

Then, we'll divide time from that.

Displacement can be found by integrating the velocity function $B(t)$ with respect to time. We will integrate it from 0 to 10 minutes.

$$\int_0^{10} (t^3 - 6t^2 + 300)dt = \frac{t^4}{4} - 2t^3 + 300t|_0^{10} = 3500 \text{ meters.}$$

Now that we have the total displacement, we must divide the total time from this to find the average velocity. The total time is 10 minutes.

Thus, the average velocity is 350 meters/minutes.

Problem — 2012 AP Calculus AB FRQ (modified)

$$\text{Find the value of } \int_0^5 x\sqrt{25 - x^2}dx$$

Solution: Whenever we see two expressions, one that is a degree higher than the other, we should think of u-substitution.

Since $25 - x^2$ has a degree of 2 (higher than the degree of x which is 1), we will try substituting $u = 25 - x^2$

We can differentiate both sides of the substitution to get $du = -2xdx$
We can divide both sides by -2 to get $xdx = -\frac{du}{2}$

Now, we will substitute these expressions into our integral: $\int_0^5 x\sqrt{25 - x^2}dx$.
We can plug in u for $25 - x^2$ and $-\frac{du}{2}$ for xdx to convert the integral to

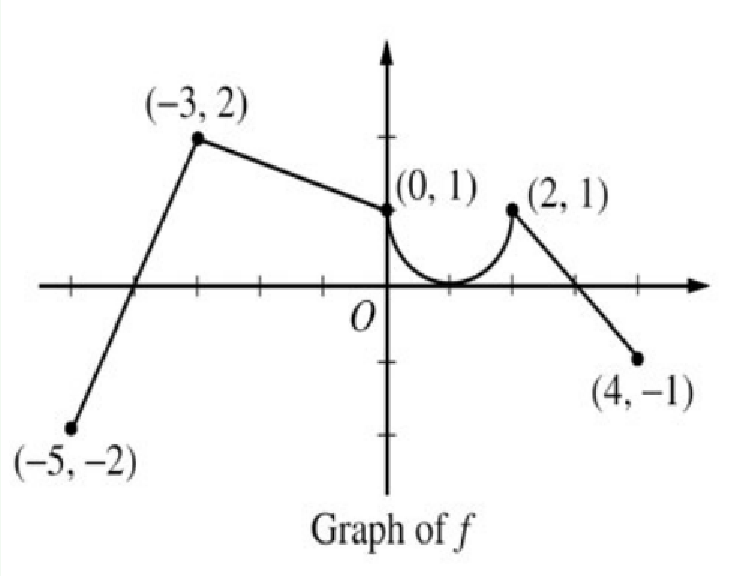
$$\int_0^5 -\frac{\sqrt{u}}{2}du$$

However, there is one more thing we must change. Our integration bounds was 0 to 5 for x. We must plug both of these numbers into $u = 25 - x^2$ to find that the new integration bounds for u are from 25 to 0.

Bringing the constant of $-\frac{1}{2}$ out and using the right integration bounds gives $-\frac{1}{2}\int_{25}^0 \sqrt{u}du$

$$\text{Evaluating gives } -\frac{1}{2}\int_{25}^0 \sqrt{u}du = -\frac{1}{2}(\frac{2}{3}u^{3/2}|_2 5^0 = -\frac{1}{2}(0 - \frac{2}{3}\cdot 25^{3/2}) = \frac{125}{3}$$

Problem — 2004 AP Calculus AB FRQ



The graph of the function f shown above consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)dt$.

(a) Find $g(0)$ and $g'(0)$.

(b) Find all values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.

Solution to part a: We know that $g(0) = \int_{-3}^{0} f(t)dt$
We can find $\int_{-3}^{0} f(t)dt$ since our area is a trapezoid. This means that

$$g(0) = \int_{-3}^{0} f(t)dt = (\frac{2+1}{2}) \cdot 3 = \frac{9}{2}$$

Now, we can find $g'(x)$ from the fundamental theorem of calculus.

We know that $g'(x) = \frac{d}{dx}[\int_{-3}^{x} f(t)dt] = f(x)$

We can plug in $x = 0$ to find that $g'(0) = f(0) = 1$

Solution to part b: To find our relative maxima, we will first find our critical points. Then, we will test the critical points to see if the sign changes from positive to negative.

To find the critical points for $g(x)$, we will find the values of x such that $g'(x) = 0$ or is undefined.

In part a, we already found that $g'(x) = f(x)$ (because of the fundamental theorem of calculus)

Now, we can look at our graph to see that $g'(x) = f(x) = 0$ when $x = -4, 1,$ and 3.
However, no sign chances occurs for $g'(x) = f(x)$ at point $x = 1$. For $x = -4$, a sign change occurs from negative to positive. This indicates a relative minimum.

A sign change indeed occurs from positive to negative when $x = 3$. This means that a relative maximum occurs at $x = 3$.

Solution to part c: An absolute minimum value can occur either at the endpoints or one of the critical points (specifically when there is a sign change from negative to positive).

We only need to test $x = -5, -4,$ and 4. We must test -5 and 4 since they are the endpoints. We test -4 because it is a critical point for which a sign change occurs from negative to positive for $g'(x) = f(x)$.

Now, we will test all 3 points to see which one is the least.
$g(-5) = \int_{-3}^{-5} f(t)dt = 0$. The area is 0 since there are 2 congruent right triangles that we must find the area of. However, one is above the x-axis while the other is below the x-axis. This means that the area of one of them must be subtracted from the other.

$g(-4) = \int_{-3}^{-4} f(t)dt = -\int_{-4}^{-3} f(t)dt = -\frac{1\cdot 2}{2} = -1$ (the area consists of a right triangle with base 1 and height 2)

$g(4) = \int_{-3}^{4} f(t)dt = -\frac{1\cdot 2}{2} + \frac{1\cdot 2}{2} + (\frac{2+1}{2}) \cdot 3 + (2 - \frac{\pi}{2}) = \frac{13-\pi}{2}$

For those wondering where the $2 - \frac{\pi}{2}$ came from, then it's because we subtracted the area of a semicircle of radius 1 from a rectangle with side lengths of 1 and 2.

Solution to part d: We already know that $g'(x) = f(x)$. We can differentiate once more to get $g''(x) = f'(x)$.

We know that a point of inflection will occur when the sign of the second derivative changes. This means we want the sign of $g''(x) = f'(x)$ to change.

Instead of seeing when the sign of $g''(x)$ changes, we can see when the sign of $f'(x)$ changes (since both functions are equivalent). This means that we need to see when $f(x)$ goes from decreasing to increasing or vice versa.

We can observe the graph to see that the sign of $g''(x) = f'(x)$ changes at $x = -3$, 1, and 2.

Problem — 2015 AP Calculus BC FRQ
Consider the function $f(x) = \frac{1}{x^2-kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \frac{k-2x}{(x^2-kx)^2}$.

(a) Let $k = 3$, so that $f(x) = \frac{1}{x^2-3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \frac{1}{x^2-4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

Solution to part a: At $x = 4$, $y = f(4) = \frac{1}{4^2-3\cdot4} = \frac{1}{4}$
Now, we will find the slope at $x = 4$. The slope is $f'(4) = \frac{3-2\cdot4}{(4^2-3\cdot4)^2} = -\frac{5}{16}$

We can use our point-slope form to write the equation of the line.
We know that $y - y_1 = m(x - x_1)$
Plugging in the point $(4, \frac{1}{4})$ and the slope $m = -\frac{5}{16}$ gives

$$y - \frac{1}{4} = -\frac{5}{16}(x - 4)$$

Solution to part b: A relative extrema only occurs at critical points for which a sign change occurs. For a relative minimum to occur at $x = 2$, the sign of $f'(x)$ must change from negative to positive. For a relative maximum to occur at $x = 2$, the sign of $f'(x)$ must change from positive to negative.

We can plug in $k = 4$ and $x = 2$ into $f'(x) = \frac{k-2x}{(x^2-kx)^2}$.
Doing so gives $f'(2) = \frac{4-2\cdot2}{(2^2-4\cdot2)^2} = 0$
Since $f'(2) = 0$, a critical point indeed occurs at $x = 2$.

Now, we can test a point slightly less than 2 such as $x = 1.9$ and one that is greater than 2 such as 2.1. Doing so tells us that the sign of $f'(x)$ changes from positive to negative at $x = 2$. Thus, a relative maximum occurs at $x = 2$.

Solution to part c: f will have a critical point at $x = -5$ if $f'(-5) = 0$.
We can plug in $x = -5$ into $f'(x) = \frac{k-2x}{(x^2-kx)^2}$.

$$f'(-5) = \frac{k+10}{25+5k} = 0$$

Solving this tells us that $k = -10$ since it will cause the numerator to be 0 (which means $f'(5)$ will be 0).

Unit 7 Differential Equations

§7.1 and 7.2 Modeling and Verifying Differential Equations

A differential equation gives us an equation representing the rate of change of a variable with respect to another.
Often, a differential equation will give you an expression for $\frac{dy}{dx}$.

Definition 7.0.1

If a problem ever states that x is **directly proportional** to y, then we can represent this as $\frac{x}{y} = k$ or $x = ky$.

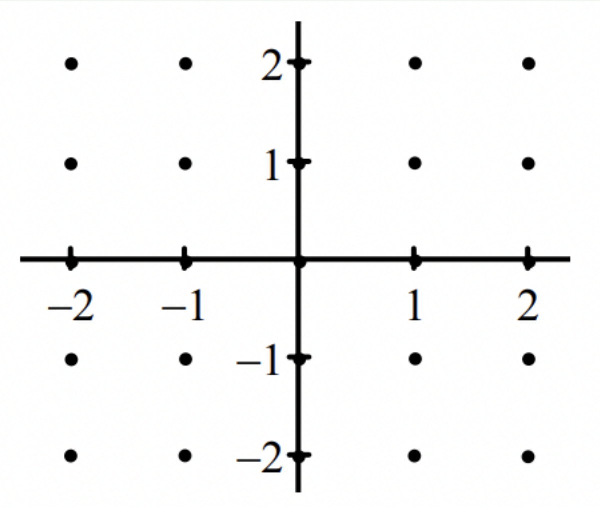
However, if a problem states that x is **inversely proportional** to y, then we can represent this as $xy = k$.
k is a constant in both scenarios above.

In addition to this, some problems might give us an equation $y(x)$ and ask us to confirm this solution for an expression given for $\frac{dy}{dx}$. In such problems, we simply plug in $y(x)$ into our differential equation and confirm if it works.

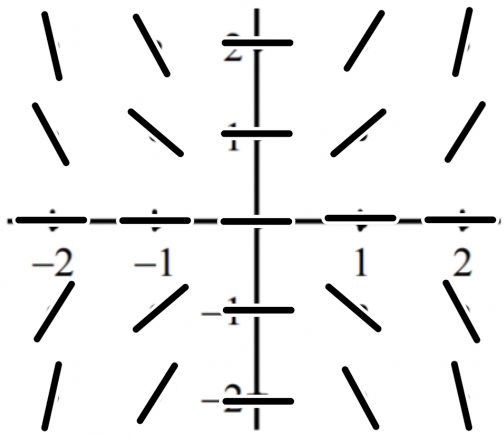
§7.3 Sketching Slope Fields

It's common to find a problem asking to graph a slope field or confirm which slpe field matches to which differential equation.
For these problems, you plug in values for x and y into an expression given for $\frac{dy}{dx}$. Then, you draw a line with that slope at that point of (x, y).

Problem — Use the differential equation $\frac{dy}{dx} = xy$ and graph its slope field below.



Solution: We'll plug in the various x, y values to find $\frac{dy}{dx}$ at that point. We'll do this for all the dotted points.
At $(0, 0)$, $\frac{dy}{dx} = xy = 0 \cdot 0 = 0$ which means the slope is 0 at that point.
At $(0, 1)$, $\frac{dy}{dx} = xy = 0 \cdot 1 = 0$ which means the slope is 0 at that point.
At $(1, 1)$, $\frac{dy}{dx} = xy = 1 \cdot 1 = 1$ which means the slope is 1 at that point.
We can do this for all of the points to graph the slope field.



Some multiple choice problems will give you a slope field and a few options with differential equations. It will ask you to determine the differential equation that will make that specific slope field.
To attack such a problem, try to think about the points for which $\frac{dy}{dx}$ will be 0 and confirm that in the graph. This can provide the most useful information and allow you to eliminate a few options.

§7.4 Reasoning Using Slope Fields

A few problems will ask you to graph a general solution on a slope field graph. You will also be given the initial point to start at.
In such a problem, you first plot the initial point. Then, you follow the slopes shown in the slope field graph and try to make the actual graph have slopes that match with the ones shown on the slope field sketch.

To see an example of this, make sure to check out the AP Calculus AB/BC Unit 7 rapid review video on the TMAS Academy youtube channel.

§7.6 General Solutions using Separation of Variables

Separation of variables also involves differential equations. The goal is to bring all terms with y and dy to one side while all terms with x and dx to the other.
Once that rearrangement is done, you integrate both sides and solve for y.

Problem — $\frac{dy}{dx} = \frac{6x^2}{y^2}$. Find $y(x)$.

Solution: Rearranging this differential equation gives $6x^2 dx = y^2 dy$
This is in our desired form since all terms with x and dx are on one side while all terms

with y and dy are on the other.

Integrating both sides gives $2x^3 + C = \frac{y^3}{3}$ (C is our constant)

Now we can multiply both sides by 3 to get $6x^3 + 3C = y^3$. HOWEVER, you must realize that $3 \cdot C$ is also a constant. We don't need to write it as $3C$. We can simply keep it as C, an arbitrary constant.

This means $6x^3 + C = y^3$.
We can take the cube root of both sides to get $y = \pm\sqrt[3]{6x^3 + C}$

Problem — $\frac{dy}{dx} = 3x(y-6)$. Find $y(x)$.

Solution: We will use separation of variables.
Rearranging the differential equation gives

$$\frac{dy}{y-6} = 3xdx$$

We can integrate both sides to get

$$\ln|y-6| = \frac{3x^2}{2} + C$$

Now we can use e as our base and consider both $\ln|y-6|$ and $\frac{3x^2}{2} + C$ to be exponents. This gives

$$e^{\ln|y-6|} = e^{\frac{3x^2}{2}+C}$$

This simplifies to $y - 6 = e^{\frac{3x^2}{2}+C}$

Now, we should write $e^{\frac{3x^2}{2}+C}$ as $e^{\frac{3x^2}{2}} \cdot e^C$. Since C is a constant, e^C is also a constant. Thus, instead of using e^C, we can simply make that our new constant C. This means $e^{\frac{3x^2}{2}+C} = Ce^{\frac{3x^2}{2}}$.

$$y = Ce^{\frac{3x^2}{2}} + 6$$

Note

For problems involving the separation of expressions with e, taking the natural log of both sides can be beneficial in many scenarios.
For problems involving expressions with $\ln x$, using that as an exponent and e as the base can be beneficial.
This may be tough to understand at first, but problems will clear it up.

Problem — $\frac{dy}{dx} = \frac{4x}{e^{3y}}$. Find $y(x)$.

Solution: We first use separation of variables.
Rearranging the differential equation gives $e^{3y}dy = 4xdx$

Integrating both sides gives $\frac{e^{3y}}{3} = 2x^2 + C$

We will multiply both sides by 3 to get $e^{3y} = 6x^2 + C$
Now, since we have a term on the left involving e, we'll take the natural log of both sides.

$\ln(e^{3y}) = \ln(6x^2 + C)$
We already know that $\ln(e^x)$ is x from our logarithm identities.
This means our equation simplifies to $3y = \ln(6x^2 + C)$.

We can divide both sides by 3 to get $y = \frac{1}{3} \cdot \ln(6x^2 + C)$.

If we want, we can apply our logarithm identities again to get $y = \ln(\sqrt[3]{6x^2 + C})$

§7.7 Finding Particular Solutions Using Separation of Variables

In the previous section, we found out about how to use separation of variables to solve a differential equation. In this section, we will do the same thing. However, we will also be given an **initial condition** which means we must substitute that initial condition into our general solution to solve for the constant C.

Problem — $\frac{dy}{dx} = (x+4)(y+7)$. Find $y(x)$ given the initial condition that $y(0) = 1$

Solution: We will use separation of variables.
Rearranging the differential equation gives $\frac{dy}{y+7} = (x+4)dx$

Integrating both sides gives $\ln|y+7| = \frac{x^2}{2} + 4x + C$

Since we have ln in this equation, it can be helpful to consider e as the base and both sides to be its exponents.

This gives $e^{\ln|y+7|} = e^{\frac{x^2}{2}+4x+C}$

Using our logarithm identities, this simplifies to $y + 7 = e^{\frac{x^2}{2}+4x+C}$.

On the right side, e^C is also a constant. Thus, we can simplify this further to $y + 7 = Ce^{\frac{x^2}{2}+4x}$.

We subtract 7 from both sides to get $y = Ce^{\frac{x^2}{2}+4x} - 7$.
Now, we substitute our initial condition into this. Our initial condition states that when $x = 0$, y will be 1.
We substitute those 2 numbers in to find C (the constant).
$1 = Ce^{\frac{0^2}{2}+4\cdot 0} - 7 = C - 7$
This means that $C = 8$. We substitute the value of our constant back to get our answer.

$$y = 8e^{\frac{x^2}{2}+4x} - 7$$

§7.8 Exponential Models with Differential Equations

This topic is an application of differential equations.
You will often be given a scenario in a word problem. The goal will be to write it out with mathematical notation and solve it using techniques shown before.

An extremely popular exponential model is when the rate of change of something is proportional to the size of something. This might be confusing at first, but an example will be shown to clarify it.

Problem — The rate of change of the population is proportional to the population. Set up a differential equation for this scenario and solve.

Solution: The rate of change of the population can be represented as $\frac{dP}{dt}$. The problem states that it is proportional to the population. This means we can equate $\frac{dP}{dt}$ to kP (where k is a contant).

$\frac{dP}{dt} = kP$
We can use separation of variables. Rearranging the equation gives $\frac{dP}{P} = (k)dt$.
We can integrate both sides to get $\ln(P) = kt + C$

Using our techniques highlighted in the previous section, we know we can write this as $e^{\ln(P)} = e^{kt+C}$
This simplifies to $P = Ce^{kt}$.

Theorem 7.0.1
$\frac{dy}{dt} = ky$ has a solution of $y = Ce^{kt}$

There are two types of models, You can either have a **growth** model or a **decay** model. A growth model will have a **positive** exponent (since it goes up overtime) while a decay model will have a **negative** exponent (since it goes down overtime).

Note
It's common to see a problem testing exponential models that involves words such as **half-life**.
That simply means half of it's lifetime has been over. For example, if a certain pencil can last 100 years, then its half-life will be 50 years. This can be very confusing to understand at first, but some practice should clear it up.

Problem —
A radioactive substance has a rate of decay that can be modeled using the equation $\frac{dy}{dt} = ky$. k is a constant and t is a variable measured in years. The half-life of this substance is 200 years. In that case, what is the value of k (the constant)?

Solution: First, we solve our differential equation.
We already know that $\frac{dy}{dt} = ky$ has a solution of $y = Ce^{kt}$.

Initially (when no time has passed), t will be equivalent to 0. This means $y = Ce^{k\cdot 0} = C$
In 200 years, only half of the substance will be left. This means $\frac{y}{2}$ of the substance will be left.
Using the solution $y = Ce^{kt}$, we can write this out as $\frac{y}{2} = Ce^{200k}$

We can solve the equation $\frac{y}{2} = Ce^{200k}$ by substituting $C = y$.

This cancels out the y and gives $\frac{1}{2} = e^{200k}$.
We can take the natural log of both sides to get $\ln(\frac{1}{2}) = \ln(e^{200k})$.

This simplifies to $\ln(\frac{1}{2}) = 200k$.
Dividing both sides gives us our value of k to be $\frac{\ln(\frac{1}{2})}{200} = -0.00346$

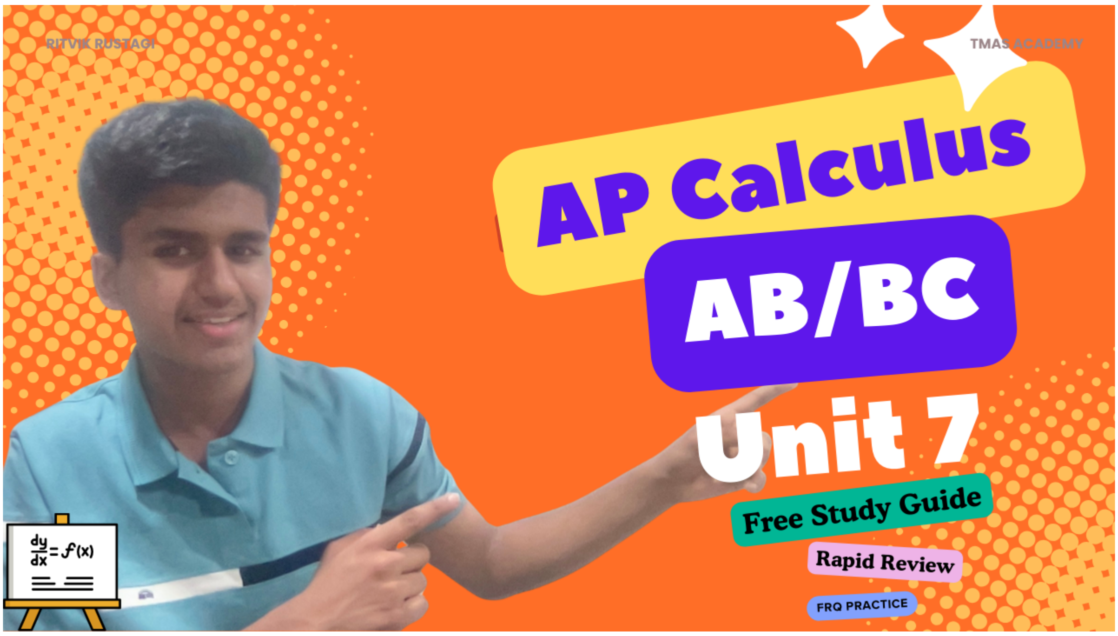
Unit 7 Practice Problems

Problem — 2016 AP Calculus AB FRQ
Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

Solution: Video Solution



Problem — 2005 AP Calculus AB FRQ
Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

Solution: We must use separation of variables to solve this differential equation.
Rearranging it gives $ydy - 2xdx$
We can integrate both sides to get $\frac{y^2}{2} = -x^2 + C$ (don't forget the constant).

Since $(1, -1)$ lies on the equation we found, we can plug in that point.
Doing so gives $\frac{(-1)^2}{2} = -1^2 + C$
We can solve for the constant C to find that $C = \frac{3}{2}$

Plugging this back into $\frac{y^2}{2} = -x^2 + C$ tells us that $\frac{y^2}{2} = -x^2 + \frac{3}{2}$
We can solve for y by first multiplying both sides by 2 and then taking the square root of both sides.
Doing so gives $y = f(x) = \pm\sqrt{-2x^2+3}$. We must decide whether the answer is $\sqrt{-2x^2+3}$ or $-\sqrt{-2x^2+3}$.

By plugging in the point $(1, -1)$, it's evident that the answer is $y = f(x) = -\sqrt{-2x^2+3}$

Problem — 2010 AP Calculus AB FRQ (modified)
Given the differential equation $\frac{dy}{dx} = xy^3$, find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

Solution: We will use separation of variables to rearrange the differential equation.

$$\frac{1}{y^3}dy = xdx$$

Now, we can integrate both sides to get $-\frac{1}{2y^2} = \frac{x^2}{2} + C$.

We can plug in our point $(1, 2)$ since it will allow us to find the integration constant C.
Doing so gives $-\frac{1}{2(2)^2} = \frac{1^2}{2} + C$. We can isolate C to find that $C = -\frac{5}{8}$.
We can plug this back in to find that

$$-\frac{1}{2y^2} = \frac{x^2}{2} - \frac{5}{8}$$

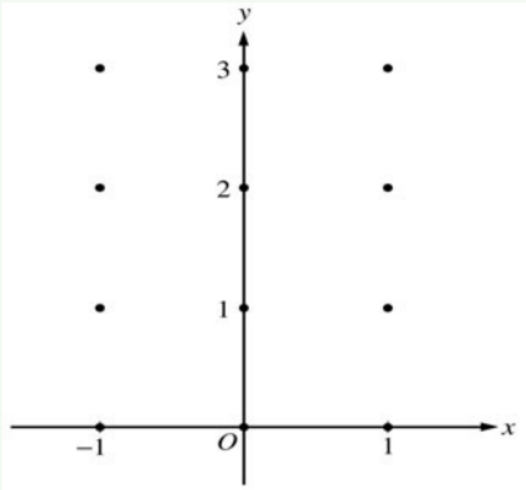
We can multiply both sides by 8 (since this cancels out all fractions on the right side) to get $-\frac{4}{y^2} = 4x^2 - 5$
We can rearrange this equation to get $-\frac{4}{4x^2-5} = y^2$ which is the same thing as $y^2 = \frac{4}{5-4x^2}$

Now, we can take the square root of both sides to find that

$$y = \frac{2}{\sqrt{5-4x^2}}$$

Also, note that there are restrictions on the value of x. The reason is that the number inside a square root must be greater than 0. Thus means that $5-4x^2>0$. We can solve this inequality to find that x must be less than $\frac{\sqrt{5}}{2}$ but more than $-\frac{\sqrt{5}}{2}$ for $y=\frac{2}{\sqrt{5-4x^2}}$

Problem — 2004 AP Calculus AB FRQ
Consider the differential equation $\frac{dy}{dx}=x^2(y-1)$

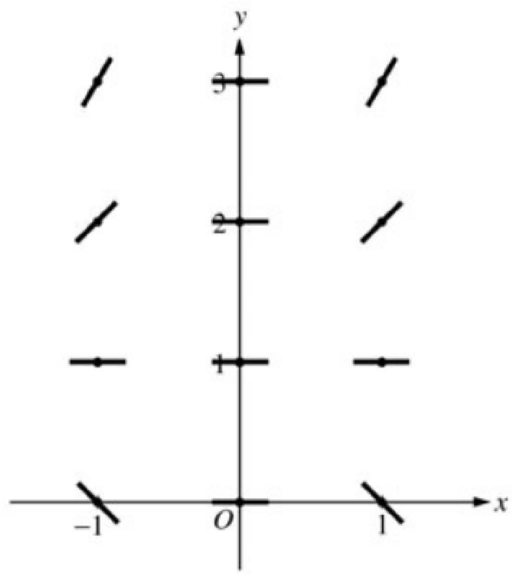


(a) On the axes provided above, sketch a slope field for the given differential equation at the twelve points indicated.

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are positive.

(c) Find the particular solution $y=f(x)$ to the given differential equation with the initial condition $f(0)=3$.

Solution to part a: To make the slope field, we simply plug in the coordinates of the certain points we want the slope for.
For example, the slope at $(0,0)$ is $\frac{dy}{dx}=0^2(0-1)=0$
We can do this for the 12 points on the graph to find the slope field.



Solution to part b: x^2 will always be positive as long as $x \neq 0$. If $x = 0$, then the entire expression $\frac{dy}{dx} = x^2(y-1)$ will be 0.
We also want $y - 1$ to be positive for $\frac{dy}{dx}$ (the slope) to be positive. Thus, y must be greater than 1.

Solution to part c: We must use separation of variables.
Rearranging the differential equation gives $\frac{dy}{y-1} = x^2 dx$
We can integrate both sides to get $\ln|y-1| = \frac{x^3}{3} + C$

This means that $y - 1 = e^{\frac{x^3}{3}+C} = Ce^{\frac{x^3}{3}}$
We can add 1 to both sides to get $y = 1 + Ce^{\frac{x^3}{3}}$

Since the point $(0, 3)$ lies on this curve, we can plug in that point to find the constant C.
Doing so gives $3 = 1 + Ce^0 = 1 + C$. This means that $C = 2$.

Thus, our solution is $y = f(x) = 1 + 2e^{\frac{x^3}{3}}$

Problem — 2011 AP Calculus AB FRQ
At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with the initial condition $W(0) = 1400$.

Solution to part a: We must first write an equation for the tangent line at time $t = 0$. To do this, we must first find the slope of the line.
We an plug in 0 for t into $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ to find that $\frac{dW}{dt} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$
Note that $W(0) = 1400$ since that's the initial amount of tons of solid waste.

Thus, the slope m for the line $y = mt + b$ is 44 (we use $y = mt + b$ instead of $y = mx + b$ since t has the same role that x would).
We can plug that in to get $y = 44t + b$.
We know that the tangent line will go through the point $t = 0$. At $t = 0$, the amount of gallons will be 1400.
This means that $1400 = 44 \cdot 0 + b$ which gives that $b = 1400$.

Thus, the equation of the tangent line is $y = 44t + 1400$.
We can plug in $t = \frac{1}{4}$ to find that $y = 44 \cdot \frac{1}{4} + 1400 = 1411$ tons (amount at the end of the first 3 months).

Solution to part b: The approximation will be an overestimate if $W''(t) < 0$ for the times between $t = 0$ and $t = \frac{1}{4}$. Similarly, it will be an underestimate if $W''(t) > 0$ for the times between $t = 0$ and $t = \frac{1}{4}$.

$$\frac{d^2W}{dt^2} = \frac{d}{dt}[\frac{dW}{dt}]$$
$$= \frac{d}{dt}[\frac{1}{25}(W - 300)] = \frac{1}{25} \cdot \frac{dW}{dt} = \frac{W - 300}{625}$$

Clearly, the initial weight was 1400 while the weight at time $t = \frac{1}{4}$ was 1411. Thus, the second derivative will be positive. We can see this by plugging in these weights into the expression we found for $\frac{d^2W}{dt^2}$

Since $\frac{d^2W}{dt^2} > 0$, our approximation for $W(\frac{1}{4})$ is an underestimate of the actual value.

Solution to part c: We must use separation of variables to rearrange our differential equation.

Doing so gives $\frac{1}{W-300}dW = \frac{1}{25}dt$
We can integrate both sides to get $\ln|W-300| = \frac{t}{25} + C$ (don't forget the integration constant C)

We can simplify the equation to get $|W-300| = e^{\frac{t}{25}+C} = Ce^{t/25}$

At time $t = 0$, we know that the weight is 1400.
We can plug in this point since it will allow us to find the constant C.
Doing so gives $1400 - 300 = Ce^{\frac{0}{25}} = C$. We can find that $C = 1100$.

Plugging this constant back in gives that $W - 300 = 1110e^{t/25}$. We can add 300 to both sides to find that $W = 300 + 1110e^{t/25}$

Problem — 2001 AP Calculus AB FRQ
The function f is differentiable for all real numbers. The point $(3, \frac{1}{4})$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6-2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $(3, \frac{1}{4})$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6-2x)$ with the initial condition $f(3) = \frac{1}{4}$.

Solution to part a: We must find an expression for $\frac{d^2y}{dx^2}$. We can do this by differentiating $\frac{dy}{dx} = y^2(6-2x)$ by using the product rule.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[\frac{dy}{dx}] = \frac{d}{dx}[y^2(6-2x)]$$
$$= y^2 \cdot \frac{d}{dx}[6-2x] + \frac{d}{dx}[y^2] \cdot (6-2x)$$
$$= y^2 \cdot -2 + 2y\frac{dy}{dx} \cdot (6-2x) = -2y^2 + 2y(6-2x)\frac{dy}{dx}$$

Using implicit differentiation, we were able to find that $\frac{d^2y}{dx^2} = -2y^2 + 2y(6-2x)\frac{dy}{dx}$. Now, we can plug in our expression for $\frac{dy}{dx}$ to simplify the expression for $\frac{d^2y}{dx^2}$ further.

$$\frac{d^2y}{dx^2} = -2y^2 + 2y(6-2x) \cdot y^2(6-2x) = -2y^2 + 2y^3(6-2x)^2$$

Now, we can plug in the point $(3, \frac{1}{4})$.

$$\text{Doing so gives } \frac{d^2y}{dx^2} = -2(\frac{1}{4})^2 + 2(\frac{1}{4})^3(6-2\cdot 3)^2 = -\frac{1}{8}$$

Solution to part b: We will use separation of variables to bring all terms with y on one side and terms with x on the other.
Doing so gives $\frac{dy}{y^2} = (6-2x)dx$

Now, we can integrate both sides to get $-\frac{1}{y} = 6x - x^2 + C$ (Don't forget to add the constant C).

We can now plug in the point $(3, \frac{1}{4})$ to find the constant C. Doing so gives

$$-4 = 6 \cdot 3 - 3^2 + C = 18 - 9 + C = 9 + C$$

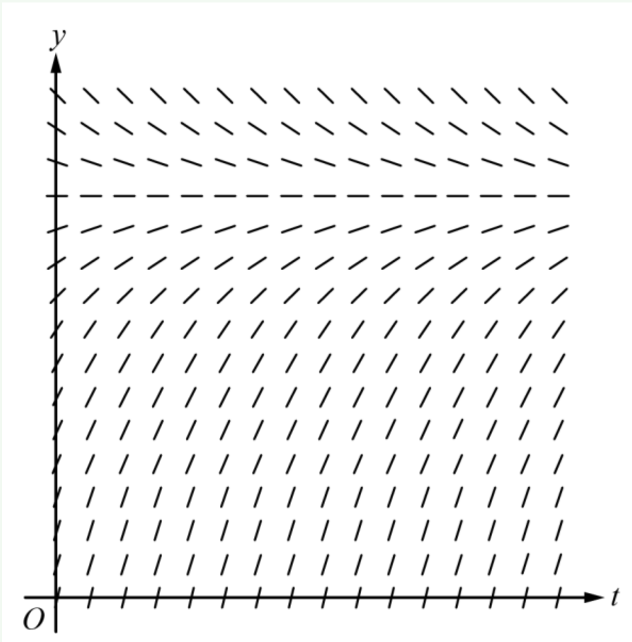
We can subtract 9 from both sides to find that $C = -13$
We can now plug this into $-\frac{1}{y} = 6x - x^2 + C$ to find that $-\frac{1}{y} = 6x - x^2 - 13$

Now, we can bring y to it's own side and rearrange to find that $y = \frac{1}{x^2-6x+13}$

Problem — 2021 AP Calculus AB FRQ
A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that sfies the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

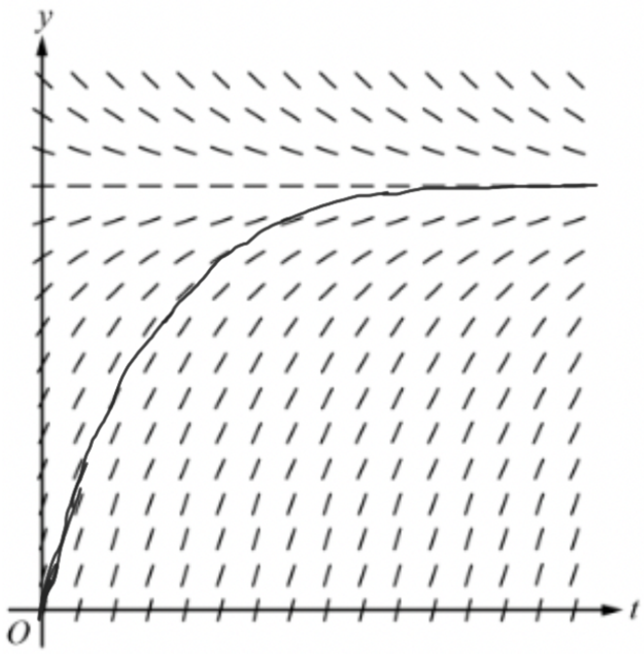
(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.

(b) Using correct units, interpret the statement $\lim_{t\to\infty} A(t) = 12$ in the context of this problem.



(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$ with initial condition $A(0) = 0$.

Solution to part a: To graph the solution curve, we first graph the initial point which is $(0, 0)$. Then, we'll try to follow the slope field lines to graph the entire solution.
It's easy to notice that the solution curve should increase and reach a limiting value due to the horizontal asymptote. The horizontal asymptote occurs when all of the slope field lines are horizontal, meaning that their slopes are 0.



Solution to part b: $\lim_{t\to\infty} A(t) = 12$ means that the amount of medication in the patient reaches 12 milligrams overtime. Since t approaches infinity, we can tell that this limit heads towards the horizontal asymptote.

Solution to part c: We will first find the general solution. Then, we will use our initial condition to find the particular solution. First, we will rearrange the equation.

$$\frac{dy}{12-y} = \frac{dt}{3}$$

Integrating both sides gives $\ln|12-y| = -\frac{t}{3} + C$.

Using e as the base, we can consider both sides of the equations to be exponents for e.

$$e^{\ln|12-y|} = e^{-\frac{t}{3}+C}$$

This simplifies to $12 - y = Ce^{-\frac{t}{3}}$

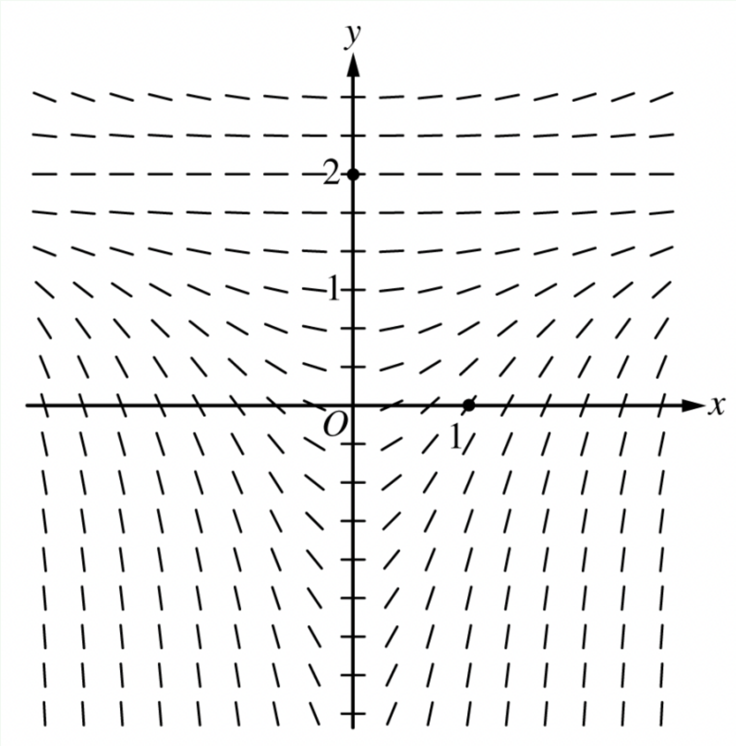
We can rearrange the equation to get $y = 12 - Ce^{-\frac{t}{3}}$, and this is our general solution.
We can find the particular solution by using the initial condition that $A(0) = 0$. We will substitute $t = 0$ and $y = 0$ into the equation to find C (the constant).

This gives $0 = 12 - C$ which means that $C = 12$.

This means our particular solution is $y = 12 - 12e^{-\frac{t}{3}}$

Problem — 2018 AP Calculus AB FRQ
Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.

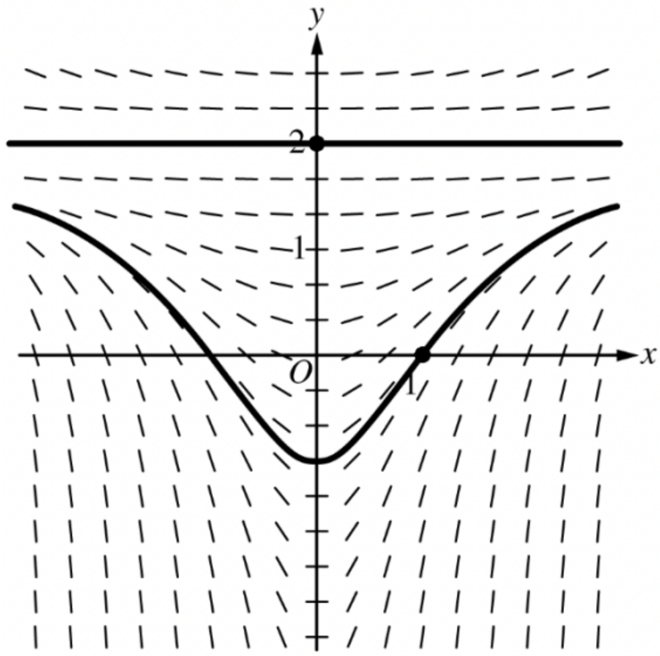


(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

Solution to part a: We can trace the solution for the graph that passes through the point $(1, 0)$ by following the slope field lines. It will allow us to graph the solution.

For the graph that goes through the point $(2, 0)$, we can notice that the slope field lines are always horizontal. This means that the solution is simply a straight line for that point.



Solution to part b: The equation of any line will be in the form of $y = mx + b$.
m represents the slope. The slope of $y = f(x)$ at $x = 1$ can be found by working with $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

Since $f(1) = 0$, we can plug in $x = 1$ and $y = 0$ into our expression for $\frac{dy}{dx}$ to find the slope of the tangent line.

$$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2 = \frac{1}{3} \cdot 1 \cdot (-2)^2 = \frac{4}{3}$$

This means our value of m is $\frac{4}{3}$, and the equation of our tangent line becomes $y = \frac{4}{3}x + b$.

Since the tangent line goes through the point $(1, 0)$, we can plug in that point into the equation to find b.
Plugging the point in gives $0 = \frac{4}{3} + b$. This means $b = -\frac{4}{3}$.
The equation for our tangent line is $y = \frac{4}{3}x - \frac{4}{3}$.

We can approximate $f(0.7)$ using this tangent line by plugging in $x = 0.7$.

$$y = f(0.7) \approx \frac{4}{3} \cdot 0.7 - \frac{4}{3} = -0.4$$

Solution to part c: We use separation of variables in this problem. We wil first rearrange our differential equation.

$$\frac{dy}{(y-2)^2} = \frac{x}{3}dx$$

Integrating both sides gives $-(y-2)^{-1} = \frac{x^2}{6} + C$

Now, we will plug in our initial condition $f(1) = 0$.
We plug in $x = 1$ and $y = 0$ into the expression to get $-(0-2)^{-1} = \frac{1^2}{6} + C$

This can be simplified further to $\frac{1}{2} = \frac{1}{6} + C$ which means $C = \frac{1}{3}$.

We can plug in our value of the constant C to get $-\frac{1}{y-2} = \frac{x^2}{6} + \frac{1}{3} = \frac{x^2+2}{6}$.

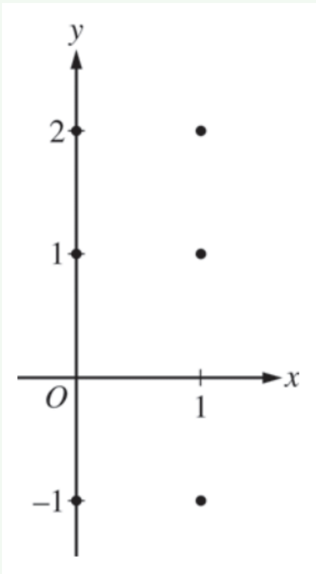
Multiplying both sides by $6(y-2)$ gives $-6 = (x^2+2)(y-2)$. Now we divide both sides by x^2+2 to get $-\frac{6}{x^2+2} = y-2$

Adding 2 to both sides gives the particular solution is

$$y = -\frac{6}{x^2+2} + 2$$

Problem — 2015 AP Calculus AB FRQ Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

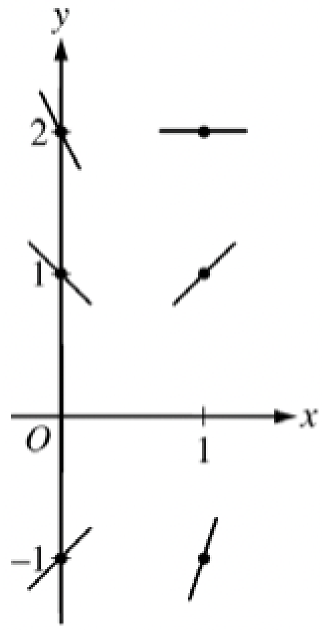


(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

Solution to part a: To graph the slope field, we will plug in the points from the grid. For example, at the point $(1, 1)$, the slope is $\frac{dy}{dx} = 2 \cdot 1 - 1 = 1$.
We can do these with the remaining points to find the slope field shown below.



Solution to part b: Since we are already given $\frac{dy}{dx} = 2x - y$, we just have to differentiate this equation once with respect to x to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x - y)$$

Using implicit differentiation, we get $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx})$

Now we can simply plug in our given $\frac{dy}{dx} = 2x - y$ into our expression for $\frac{d^2y}{dx^2}$.

$$\text{This gives } \frac{d^2y}{dx^2} = 2 - (2x - y) = 2 - 2x + y$$

In Quadrant 2, x will always be negative while y will be positive.
This means that the sign of $\frac{d^2y}{dx^2}$ will also be positive because the term $-2x$ becomes a positive number since -2 (a negative number) is multiplied to another negative number (x).

Since $\dfrac{d^2y}{dx^2}$ is positive in Quadrant 2, all solution curves in that quadrant are **concave up**

Solution to part c: A relative minimum or maximum can only occur at a critical point. A critical point is any point for which the first derivative is 0 or undefined.

There will be a relative extrema at $x = 2$ if $\frac{dy}{dx}$ is 0 at that point.
Since $y = f(x)$, we immediately know that $y = f(2) = 3$. This means we find the value of $\frac{dy}{dx}$ when $x = 2$ and $y = 3$.

$$\frac{dy}{dx} = 2 \cdot 2 - 3 = -1$$

Since the first derivative is equal to -1, a critical point can't occur at this point. Thus, the answer is that **neither** exists because the first derivative at $x = 2$ isn't 0, which means that it can't be a critical point. Since it's not a critical point, there will neither be a relative minimum nor a relative maximum at that point since all relative extrema must occur at critical points.

For those that are confused about this part, make sure to check out Unit 5 where we discussed all the different tests such as the first and second derivative tests.

Solution to part d: In many problems that involve finding the particular/general solution, many people immediately jump to separation of variables. However, in this problem you use the form given in the problem statement. We will work backwards as we are already given the solution to the differential equation

Since we know that our solution is in the form $y = mx + b$, we can plug this equation into the differential equation.

$$\text{We know that } \frac{dy}{dx} = 2x - y$$

Plugging in $y = mx + b$ into this gives $\frac{d}{dx}(mx + b) = 2x - (mx + b)$.

Simplifying this gives $m = 2x - mx - b$.
Subtracting m from both sides gives $2x - mx - m - b = 0$. This can be factored as $x(2 - m) - (m + b) = 0$.

This equation means that $2 - m = 0$ and $m + b = 0$.
Solving $2 - m = 0$ gives that $m = 2$ which means that $b = -2$.
By simply plugging in $y = mx + b$ into the differential equation, we were able to find that $m = 2$ and $b = -2$.

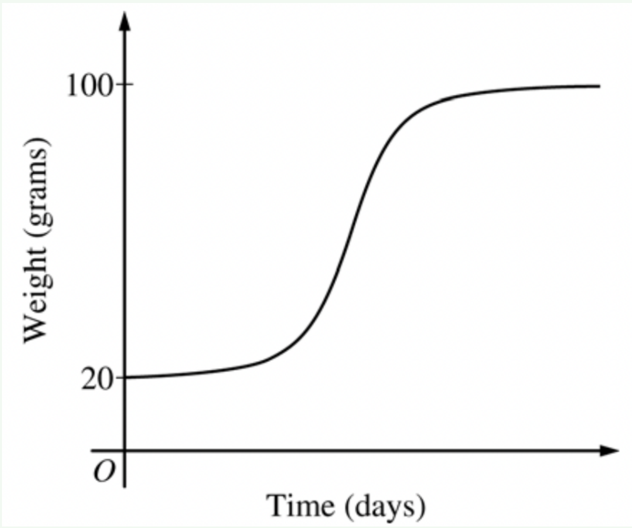
Problem — 2012 AP Calculus BC FRQ
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weights 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

Solution to part a: To check whether the bird is gaining weight faster when it weighs either 40 or 70, we find the value of $\frac{dB}{dt}$ for each weight. The one that's the highest shows the greatest rate of change.

$$\text{When the weight is 40 grams: } \frac{dB}{dt} = \frac{1}{5} \cdot (100 - 40) = \frac{1}{5} \cdot 60 = 12$$
$$\text{When the weight is 70 grams: } \frac{dB}{dt} = \frac{1}{5} \cdot (100 - 70) = \frac{1}{5} \cdot 30 = 6$$

Clearly the bird gains weight faster when it weights 40 grams since $\frac{dB}{dt}$ has a higher value at a weight of 40 than 70. A greater $\frac{dB}{dt}$ value means that the weight is increasing faster due to the higher rate of change.

Solution to part b: To find $\frac{d^2B}{dt^2}$, we simply take the derivative of $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

with respect to t once.

$$\frac{d^2B}{dt^2} = \frac{d}{dt}(\frac{dB}{dt}) = \frac{d}{dt}(\frac{1}{5}(100 - B)) = -\frac{1}{5} \cdot \frac{dB}{dt}$$

Now, we plug in our expression for $\frac{dB}{dt}$ to find $\frac{d^2B}{dt^2}$.

$$\frac{d^2B}{dt^2} = -\frac{1}{5} \cdot (100 - B) = -20 + \frac{B}{5}$$

$\frac{d^2B}{dt^2}$ is always concave down for any weight less than 100. This can be observed from our expression for $\frac{d^2B}{dt^2}$ since the second derivative will be negative when $B < 100$.

However, our graph for weight and time shows that the curve is concave up for some parts of the graph. This contradicts what we found; we found that $\frac{d^2B}{dt^2}$ must be negative for the portion shown $(20 \leq B < 100)$, meaning that the graph of the weight with respect to time should be concave down.

Solution to part c: We can rearrange the differential equation to get $\frac{dB}{100-B} = \frac{dt}{5}$.

Integrating both sides gives $-\ln|100 - B| = \frac{t}{5} + C$

We can divide both sides by -1 to get $\ln|100 - B| = -\frac{t}{5} + C$

We can use e as the base and both sides as the exponent to get $e^{\ln|100-B|} = Ce^{-\frac{t}{5}}$

This simplifies to $100 - B = Ce^{-\frac{t}{5}}$

Since the initial condition is $B(0) = 20$, we cab substitute $t = 0$ and $B = 20$ to find the constant C.

Doing so gives that $C = 80$. We can plug this in to get $100 - B = 80e^{-\frac{t}{5}}$.
Rearranging both sides gives that our particular solution is

$$B = 100 - 80e^{-\frac{t}{5}}$$

Problem — 2013 AP Calculus AB FRQ
Consider the differential equation $\frac{dy}{dx} = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

Solution to part a: To find the equation of the tangent line, we must first find the slope at the point $(1, 0)$.

We know that $\frac{dy}{dx}$ for the point $(1, 0)$ is $e^0(3(1)^2 - 6 \cdot 1) = -3$

We can plug this in for m (the slope) to $y = mx + b$ to get $y = -3x + b$.

We know that the tangent line passes through the point $(1, 0)$. Thus, we plug that point into the line to get $0 = -3 \cdot 1 + b$. Solving for b gives that $b = 3$.

This means that the equation of the tangent line is $y = -3x + 3$.
We can plug in $x = 1.2$ to find that

$$f(1.2) \approx -3 \cdot 1.2 + 3 = -0.6$$

Solution to part b: To find the particular solution, we must use separation of variables. We can first rearrange the differential equation to get

$$\frac{1}{e^y} dy = (3x^2 - 6x)dx$$

We can integrate both sides to get $-\dfrac{1}{e^y} = x^3 - 3x^2 + C$

We know that the point $(1, 0)$ must lie on the equation we just found. Thus, we plug that point in to get

$$-\frac{1}{e^0} = 1^3 - 3(1)^2 + C = -2 + C$$

We can solve for C to find that $C = 1$
Plugging this integration constant back gives $-\frac{1}{e^y} = x^3 - 3x^2 + 1$.

We can multiply both sides by -1 to get $\frac{1}{e^y} = -x^3 + 3x^2 - 1$

Now, we can take the natural log of both sides to get $-y = \ln(-x^3 + 3x^2 - 1)$

Now, we can multiply both sides by -1 to find that

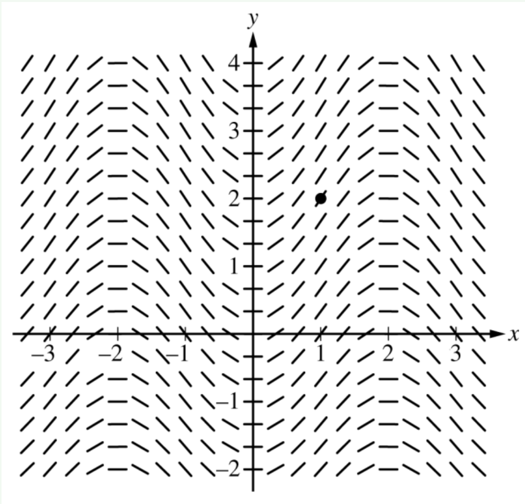
$$y = -\ln(-x^3 + 3x^2 - 1)$$

Note that $-x^3 + 3x^2 - 1$ must be greater than 0 because you can't take the natural logarithm of a negative number nor 0.

Problem — 2022 AP Calculus AB FRQ

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin(\frac{\pi}{2}x)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.

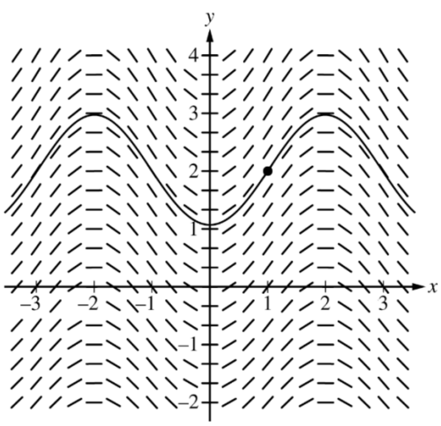


(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin(\frac{\pi}{2}x)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

Solution to part a: We plot the point $(1, 2)$ on the coordinate plane. That is the initial point we start with. Then, we try our best to trace along the slope field so that it connects to the point $(1, 2)$ while maintaining the slope for that point. Doing so will give us the solution curve below.



Solution to part b: To write the equation of the tangent line to the point $f(1,2)$, we will first find the slope at that point.
We know that $\frac{dy}{dx} = \frac{1}{2}\sin(\frac{\pi}{2} \cdot 1)\sqrt{2+7} = \frac{3}{2}$

We found that the slope will be $\frac{3}{2}$ at the point $(1,2)$. The general equation of any line is $y = mx + b$. We can plug in $\frac{3}{2}$ for m (representing the slope).

Doing so gives $y = \frac{3}{2}x + b$. We know that the tangent line passes through the point $(1,2)$. Thus, we plug the point into the equation. Doing so gives $2 = \frac{3}{2} \cdot 1 + b$

We can solve the equation above to find that $b = \frac{1}{2}$.
We an plug this back into the equation for the tangent line to get

$$y = \frac{3}{2}x + \frac{1}{2}$$

Now, we plug in $x = 0.8$ to approximate the value of $f(0.8)$. Doing so gives

$$y = f(0.8) \approx \frac{3}{2} \cdot 0.8 + \frac{1}{2} = 1.7$$

Solution to part c: Since $f''(x) > 0$, f is concave up for that interval. That means the tangent line lies below the graph at $x = 0.8$. Since the tangent line lies below the graph, the value we found for $f(0.8)$ using the tangent line is an underestimate.

Solution to part d: We rearrange the differential equation to bring all the terms with x and dx on one side and the terms with y and dy on the other.

Doing so gives $\frac{dy}{\sqrt{y+7}} = \frac{1}{2}\sin(\frac{\pi}{2}x)dx$
We can integrate both sides to get $2\sqrt{y+7} = -\frac{1}{\pi}\cos(\frac{\pi}{2}x) + C$

We know the point $(1,2)$ lies on this particular solution. Thus, we plug that point in to get $2\sqrt{2+7} = -\frac{1}{\pi}\cos(\frac{\pi}{2}) + C$

The above expression simplifies to $6 = -\frac{1}{\pi} \cdot 0 + C$

That means $C = 6$. We can plug that in to find that

$$2\sqrt{y+7} = -\frac{1}{\pi}\cos(\frac{\pi}{2}x) + 6$$

Now, we can divide both sides by 2 to get

$$\sqrt{y+7} = -\frac{1}{2\pi}\cos(\frac{\pi}{2}x) + 3$$

We can square both sides and then subtract 7 from both sides to get

$$y = (-\frac{1}{2\pi}\cos(\frac{\pi}{2}x) + 3)^2 - 7$$

Problem — 2016 AP Calculus BC FRQ
Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation $g(-1) = 2$.

$$\text{Find} \quad \lim_{x\to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right).$$

Show the work that leads to your answer.

Solution to part a: We can find $\frac{d^2y}{dx^2}$ by differentiating the expression for $\frac{dy}{dx}$ with respect to x.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(x^2 - \frac{1}{2}y) = 2x - \frac{1}{2}\cdot\frac{dy}{dx}$$

We can simplify the expression above further by substituting $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ into it.

$$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}(x^2 - \frac{1}{2}y)$$

Solution to part b: A relative minimum exists whenever $f'(c) = 0$ and $f''(c) > 0$. Similarly, a relative maximum exists whenever $f'(c) = 0$ and $f''(c) < 0$.

We will first test to see whether or not $\frac{dy}{dx} = 0$ at the point $(-2, 8)$ which will prove whether or not its a critical point.

Plugging in $x = -2$ and $y = 8$ gives $\frac{dy}{dx} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$.

Since a critical point exists at $x = -2$, we now only need to check whether $\frac{d^2y}{dx^2}$ is less than 0 or greater than 0. This is part of the Second Derivative Test. If you're confused about these steps, make sure to go back and check out that test from Unit 5.

We can plug in $x = -2$ and $y = 8$ into $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}(x^2 - \frac{1}{2}y)$ to get that $\frac{d^2y}{dx^2} = -4$. Since the second derivative is negative (meaning that its < 0), a **relative maximum** occurs at the point $(-2, 8)$.

In summary, the graph of f has a relative maximum at the point $(-2, 8)$ because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

Solution to part c: We will first try direct substitution with our limit.

$$\lim_{x\to -1}\frac{g(x) - 2}{3(x+1)^2} = \frac{g(1) - 2}{3\cdot 0^2} = \frac{0}{0}$$

Since our limit evaluates to an indeterminate form, we should immediately think about L'Hopitals Rule.
Applying L'Hopitals rule gives

$$\lim_{x\to -1} \frac{g(x)-2}{3(x+1)^2} = \lim_{x\to -1} \frac{g'(x)}{6(x+1)}$$

Now we try direct substitution on $\lim_{x\to -1} \frac{g'(x)}{6(x+1)}$. We get an indeterminate form $\frac{0}{0}$ again. Thus, we apply L'Hopitals Rule again.

$$\lim_{x\to -1} \frac{g'(x)}{6(x+1)} = \lim_{x\to -1} \frac{g''(x)}{6}$$

To find $\lim_{x\to -1} \frac{g''(x)}{6x}$, we first find $g''(-1)$. We find the second derivative at the point $(-1, 2)$. Plugging in that point into our expression for $\frac{d^2y}{dx^2}$ in part a gives -2.

We directly substitute this into $\lim_{x\to -1} \frac{g''(x)}{6}$ to get that the limit is $-\frac{1}{3}$.

Unit 8 Applications of Integration

§8.1 Average Value of a Function

The average value of a function over an interval is the sum of all possible values in that interval divided by the interval width. The sum of all possible values is hard to find since there are an infinite number of points in an interval (since a point can have as many decimal points as possible). That is why an integral is used to find the average value over an interval.

Theorem 8.0.1

The average value of a function $f(x)$ on an interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

$\int_a^b f(x)$ represents the sum of all possible values while $b-a$ is the interval length.

Problem — Find the average value of $f(x) = \frac{1}{x^2}$ over the interval $[3, 6]$.

Solution: We apply the formula that states that the average value of a function over the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Applying that to this problem gives $\frac{1}{6-3}\int_3^6 f(x)dx = \frac{1}{3}\int_3^6 \frac{1}{x^2}dx$

$$\frac{1}{3}\int_3^6 \frac{1}{x^2}dx = \frac{1}{3}(-\frac{1}{x}|_3^6) = \frac{1}{3}(-\frac{1}{6} - (-\frac{1}{3}) = \frac{1}{18}$$

§8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals

This topic connects some of our variables that often show up on the AP Calculus AB and BC exam.
You should know the two relationships below.
$v(t) = \int a(t)dt$ (the integral of acceleration with respect to time is the velocity function).
$x(t) = \int v(t)dt$ (the integral of velocity with respect to time is the position function).

Note

How would you find an expression for the total **distance** travelled when the velocity function is $v(t)$?

This is where most people mess up. Distance is a quantity that does not depend on direction. It is a **scalar** quantity. It ONLY depends on magnitude.
If someone said that the distance travelled is $\int v(t)dt$, then they are completely wrong.
The reason is that velocity can be negative. However, when finding distance, speed is used, and we know that speed only has magnitude (not a direction). We can find our distance by integrating the absolute value of $v(t)$.
The **distance** travelled is $\int |v(t)|dt$

Problems for this section will be shown at the end of this unit.

§8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts

This section doesn't have any theory. It's just about practicing a few problems.
You need to know the context the problem is happening in. This just involves reading the description. For example, if a problem states that $f(x) = \frac{dP}{dt}$ represents the change in population with respect to time, then $\int f(x)dx$ represents the population.

Accumulation functions simply measure the area under a graph. If we know the rate of change of something, then integrating over a certain interval can tell us the amount that it changed.

Problem — 2010 AP Calculus AB FRQ
There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 a.m., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 a.m. ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0, & \text{if } 0 \leq t < 6 \\ 125, & \text{if } 6 < t < 7 \\ 108, & \text{if } 7 \leq t \leq 9 \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 a.m.?

(b) Find the rate of change of the volume of snow on the driveway at 8 a.m.

Solution to part a: $f(t)$ represents the rate at which snow accumulates. Thus, we can integrate $f(t)$ to find the total amount of snow that accumulates. We'll use our calculator to find the answer to the integral.

$$\int_0^6 f(t)dt = \int_0^6 7te^{\cos t}dt = 142.275 \text{ ft}^3$$

Solution to part b: At 8 a.m., not only does snow accumulate, but it's also being removed by Janet.

$$\text{The rate of change of snow is } f(t) - g(t)$$

$f(t)$ is the rate at which snow accumulates and $g(t)$ is the rate at which it is removed.

The rate at which the volume changes is simply $f(8)-g(8)$ (the difference in the two rates).

$$\text{We can find that } f(8) - g(8) = 7 \cdot 8e^{\cos 8} - 108 = -59.583 \text{ ft}^3/\text{hour}$$

§8.4 Area Between Curves with respect to x

Before simply memorizing the formula for the area between a curve with respect to x, it's important to visualize it.
When finding an area between curves with respect to x, we make infinite, thin, vertical slices. The area will simply be sum of all those slices.

Assuming our two curves that we want to find the area between are $f(x)$ and $g(x)$, then the height of one slice is $f(x) - g(x)$. The width of it is dx (some very small amount).
That means the area of that slice is $[f(x) - g(x)]dx$.
In general, the function that you subtract should be the one that's below the other equation. This might be confusing to understand, but a few examples should clarify it.

Now the final problem is to figure out the bounds that should be used when integrating for the area between curves. If a problem explicitly states to use certain bounds, then you should those bounds.

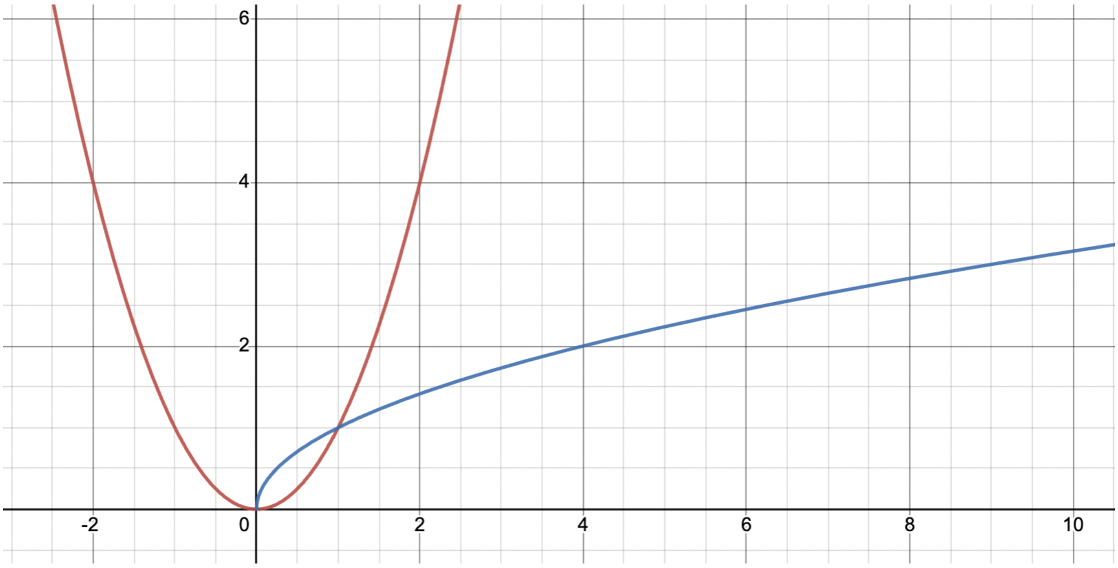
However, some problems will expect you to find the bounds on your own. To find the area between two curves, you can set them equal to each other. Then, you can solve for the intersection points and find the integral by using the intersection points as the bounds.

Note

To find the area between curves with respect to x, you should follow these few steps.
1. Graph the two curves. This is extremely important to see what function lies above the other.
2. Find the intersection points of the curves if the bounds aren't given.
3. If my two curves are $f(x)$ and $g(x)$ and $f(x)$ lies above $g(x)$ for a certain interval, then I should set up the expression $\int [f(x) - g(x)]dx$. Assuming my intersection points are $x = a$ and $x = b$, I can write the integral as $\int_a^b [f(x) - g(x)]dx$

Problem — Find the area between the curves $f(x) = x^2$ and $g(x) = \sqrt{x}$.

Solution: Many people will immediately jump to setting up the integral $\int (f(x)-g(x))dx$. However, that can lead to errors. We should first **graph** the two curves.



Now, we find the intersection points for the two curves by setting $f(x)$ equal to $g(x)$.
$x^2 = \sqrt{x}$
After squaring both sides and solving, we get that $x = 0$ and $x = 1$.

Clearly, between $x = 0$ and $x = 1$, $g(x)$ lies above $f(x)$. We think about making infinite slices of height $g(x) - f(x)$. The width of each will be dx.
We can simply set up the integral $\int_0^1 (\sqrt{x} - x^2)dx$

$$\text{From the fundamental theorem of calculus, } \int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}|_0^1 = \frac{1}{3}$$

This method works when our equation is in terms of x. However, what will we do when our function is in terms of y. What if it is represented as $x = f(y)$ instead of our traditional $y = f(x)$

§8.5 Area Between Curves with respect to y

Area between curves with respect to y involve the exact same techniques as the last section.
Instead of having two functions $f(x)$ and $g(x)$, our two functions will be $f(y)$ and $g(y)$. They will be in terms of y.

Also, instead of making infinite vertical slices, now we will make infinite **horizontal** slices.
On top of that, now our width for each slice will be dy instead of dx.
This means our integral for the area between curves with respect to y can be represented as $\int [f(y) - g(y)]dy$.

Note

To find the area between curves with respect to y, you should follow these few steps (they are similar to the steps shown in the last section).

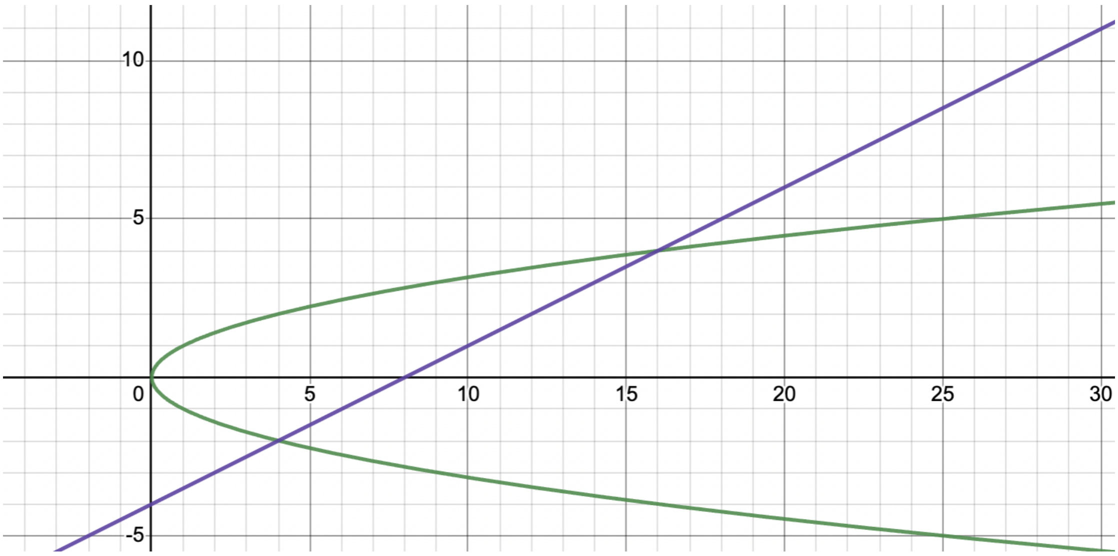
1. Graph the two curves. This is extremely important to see what function lies **to the right** of the other. Traditionally, the one that lies to the left should be subtracted.

2. Find the intersection points of the curves if the bounds aren't given.

3. If my two curves are $f(y)$ and $g(y)$ and $f(y)$ lies to the right of $g(y)$ for a certain interval, then I should set up the expression $\int[f(y) - g(y)]dy$. Assuming my intersection points are $y = a$ and $y = b$, I can write the integral as $\int_a^b [f(y) - g(y)]dy$.

Problem — What is the area between the curves $f(y) = y^2$ and $g(y) = 2y + 8$?

Solution: First, we will graph both functions to see which one lies more to the right and to help us visualize the problem.



Now, we'll find the intersection points for both graphs. We simply set $f(y)$ and $g(y)$ equal to each other.
This gives that $y^2 = 2y + 8$
Rearranging this gives $y^2 - 2y - 8$ which is a quadratic that can be factored as $(y+2)(y-4)$.
Clearly, the solutions are $y = -2$ and $y = 4$ which are the intersection points.

For the interval between $y = -2$ and $y = 4$, it's obvious that $g(y) = 2y + 8$ lies more to the right.
This means we can set up our integral as

$$\int_{-2}^{4} [g(y) - f(y)]dy = \int_{-2}^{4} [2y + 8 - y^2]dy$$

Using the fundamental theorem of calculus, we find that

$$\int_{-2}^{4} [2y + 8 - y^2]dy = y^2 + 8y - \frac{y^3}{3}|_{-2}^{4}$$

$$y^2 + 8y - \frac{y^3}{3}|_{-2}^{4} = 4^2 + 8 \cdot 4 - \frac{4^3}{3} - ((-2)^2 + 8 \cdot -2 - \frac{(-2)^3}{3}) = 36$$

§8.6 Finding the Area Between Curves That Intersect at More Than Two Points

In some problems, the curve might intersect at multiple points. Then, you'll find the area separately between each intersection point. You will often also have to set up multiply integrals and add each up separately.

Note

To find the area between curves that have multiple intersection points, you should follow these few steps (they are similar to the steps shown in the previous two sections).
1. Graph the two curves. This is extremely important to see what function lies **to the right** of the other or *above* the other. The one that lies to the left/below the other will be subtracted.

2. Find the intersection points of the curves.

3. Set up multiple integrals if necessary. In many cases, one function that is to the right/above another for a certain bound might not have that same condition apply to it for another bound. This means that if $f(x)$ lies above $g(x)$ for $a < x < b$, then that might not be the case for $b < x < c$.

4. Evaluate all of those integrals and add them up.

This might be confusing at first, but practice at the end of this unit will clear it up.

§8.7 Volumes with Cross Sections: Squares and Rectangles

So far we have found the area between two curves.

Now, we will expand our knowledge on top of that. We will have **cross-sections** perpendicular to either the x-axis or y-axis.
To do this, you will first find the area of the cross section. Each cross section will have an infinitesimally small width/depth (either dx or dy).

In simple words, to find the volume of a solid with a known cross section, we break our $3d$ object into an infinite amount of $2d$ objects with the known cross section.

Note

To find the volume with a known cross section such as a square or a rectangle, read through the steps below to make sure you never lose a point on this topic.

1. If the problem says your cross sections are perpendicular to the x-axis, then you will be integrating with dx. This means you want to make your functions be in the form $y = f(x)$ rather than $x = f(y)$.
If the problems says that the cross sections are perpendicular to the y-axis, then you will be integrating with dx. You must make sure that the functions are in the form $x = f(y)$ rather than $y = f(x)$.

2. Now, you find the area of one cross section. You first find the length of the base (the width) for each cross section (the base of the cross section. The length of the base for a cross section is the difference between the two given functions. For example, if you are told to find the volume for a problem involving functions $f(x)$ and $g(x)$, then the length of a certain base is $f(x) - g(x)$.

3. To find the height of this certain cross section, we must first determine the shape of the cross section. If it's a square, then the area is simply the base times the base (because all sides are equal in a square.

This means if our base is $f(x) - g(x)$, then the area is simply $[(f(x) - g(x)]^2$.

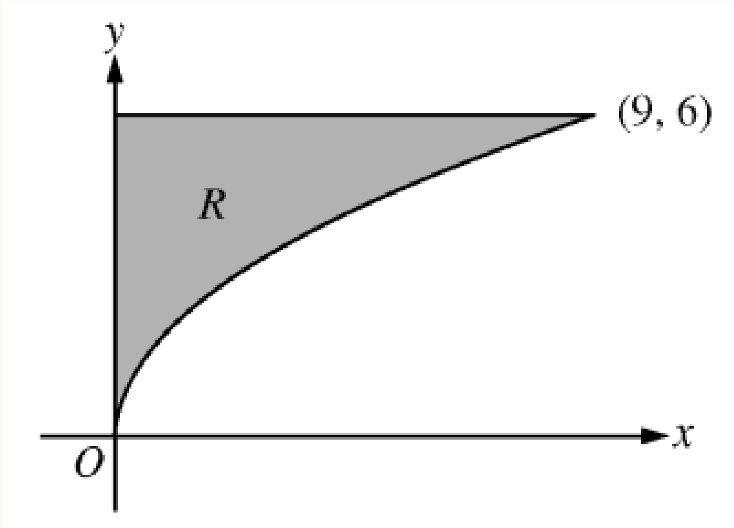
If our cross section is a rectangle, then the problem will describe the relationship between the width and the height. For example, if it says that the height is 2 times the width, then our height can be represented as $2 \cdot (f(x) - g(x))$. That means our area of the cross section for this example is simply $2[f(x) - g(x)]^2$

4. Now that we found the area of a cross section in general, we must multiply the area by its length. The length will be dx if our cross sections are perpendicular to the x-axis. It will be dy if the cross sections are perpendicular to the y-axis.

5. If bounds aren't given for integration, then we can equate the two given functions again. The values we find will be used for the bounds.

This topic can best be explained through an example problem, so let's jump into a good problem.

Problem — 2010 AP Calculus AB FRQ



Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$ and the y-axis, as shown in the figure above.
(a) Find the area of R.

(b) Part b has been skipped since it involves topics that will be covered soon later in this unit.

(c) Region R is the base of a solid. For each y, where $0 \leq y \leq 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

Solution to part a: This part is a review of section 8.4 and 8.5.
We will find the area through infinite **vertical** strips. In this case, $f(x) = 6$ and $g(x) = 2\sqrt{x}$.
The area will involve multiplying the length of each strip to the width dx. We will integrate that from $x = 0$ and $x = 9$.

We can set up our integral as $\int_0^9 [f(x) - g(x)]dx = \int_0^9 [6 - 2\sqrt{x}]dx$
We can apply the fundamental theorem of calculus to get

$$\int_0^9 [6 - 2\sqrt{x}]dx = (6x - \frac{4}{3}x^{\frac{3}{2}})|_0^9 = 18$$

Part b has been skipped since it involves topics that will be covered soon later in this unit.

Solution to part c: Now this part involves our concepts regarding cross sections creating a 3D figure.

Since our cross sections are taken perpendicular to the y-axis, we will be integrating with dy. This means we want our area for each cross section to be in terms of y.

To find the area of a rectangular cross section, we will first find the length of the base. We should first represent each equation in terms of y.

We can rewrite the equation $y = 2\sqrt{x}$ in the form $x = f(y)$ by first squaring both sides. That gives $y^2 = 4x$. Then, we can divide both sides by 4 to get $x = \frac{y^2}{4}$.

The length of the base/width of the cross section is simply the difference between $x = \frac{y^2}{4}$ and y-axis which is just $x = 0$. We can write the width as a function $w(y) = \frac{y^2}{4}$

The height can also be represented as a function. Since it is 3 times the length of the base, $h(y) = 3 \cdot w(y) = 3 \cdot \frac{y^2}{4}$.

The area of a rectangular cross section can be modelled as $A(y) = w(y) \cdot h(y) = \frac{y^2}{4} \cdot \frac{3y^2}{4}$ which is $\frac{3y^4}{16}$.

We are also supposed to integrate the area of this cross section with respect to y to find the volume. The bounds we integrate between are $y = 0$ and $y = 6$.

We can set up our integral for the volume as $\int_0^6 A(y) = \int_0^6 \frac{3y^4}{16}$. We will go a step further and solve it (on the real AP exam do not evaluate the expression as the problem clearly says to not evaluate it).

§8.8 Volume with Cross Sections: Triangles and Semicircles

The volume of our figure will still be either $\int_a^b A(x)dx$ or $\int_a^b A(y)dy$ depending on what axis our cross sections are perpendicular to.
$A(x)$ and $A(y)$ both represent the area of a cross section.

The triangles that we will often see as our cross sections include the isosceles right triangle and equilateral triangle.

If our cross section is an isosceles right triangle, we must first find the area of a generic isosceles right triangle.
One of the legs will have a length of $f(x) - g(x)$. This means that the other leg (the height) will also have the same length since an isosceles right triangle has 2 legs with the same length.
Since both sides lengths are $f(x) - g(x)$, the area of this isosceles right triangle is

$$\frac{[f(x) - g(x)]^2}{2}$$

This isosceles right triangle will have a thickness of dx (if cross sections are perpendicular to the x-axis.

Our integral can be written as $\int_a^b \frac{[f(x)-g(x)]^2}{2} dx$. This is the volume of a figure with cross sections of an isosceles triangle when all cross sections are perpendicular to the x-axis.
Similarly, if the isosceles triangle cross sections are perpendicular to the y-axis, then the volume is $\int_a^b \frac{[f(y)-g(y)]^2}{2} dy$.

We can similarly find the volume when all cross sections are an equilateral triangle. If the base is $f(x) - g(x)$, then the area will be $\frac{\sqrt{3}}{4}[f(x) - g(x)]^2$ since the area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$ where s represents the side length of an equilateral triangle.

If each equilateral triangle cross section has a thickness of dx (when cross sections are perpendicular to the x-axis), then the volume can be written as $\int_a^b \frac{\sqrt{3}}{4}[f(x) - g(x)]^2 dx$. Similarly, if the cross sections are perpendicular to the y-axis, then the volume can be written as $\int_a^b \frac{\sqrt{3}}{4}[f(y) - g(y)]^2 dy$.

Similarly, if all cross sections are a semicircle, then we already know that the length of the diameter will be $f(x) - g(x)$. This means that the radius has a length of $\frac{f(x)-g(x)}{2}$. This means the area of the semicircle can be written as $\frac{\pi[f(x)-g(x)]^2}{8}$ since the area of a semicircle is half of the area of a circle.

If each semicircle cross section has a thickness of dx, then the volume can be written as $\int_a^b \frac{\pi[f(x)-g(x)]^2}{8} dx$. If each cross section is perpendicular to the y-axis, then the volume can similarly be written as $\int_a^b \frac{\pi[f(y)-g(y)]^2}{8} dy$

Note

In general, don't forget to graph both functions since it allows us to visualize the problem easily. It will make finding the area of the cross section much easier.

Note

If a problem states that the cross sections are taken perpendicular to the x-axis, then make sure that the entire integral is written in terms of x. This can be done by rewriting all functions in the problem in the form $y = f(x)$.

Similarly, if a problem states that the cross sections are taken perpendicular to the y-axis, then make sure that the entire integral is written in terms of y which can be done by rewriting all functions in the form $x = f(y)$.

This is extremely important because when the cross sections are perpendicular to the x-axis, then you will integrate with respect to x (dx). However, when the cross sections are perpendicular to the y-axis, then you will integrate with respect to y (dy).

Problem — 2007 AP Calculus AB FRQ
Let R be the region in the first and second quadrants bounded above by the graph of $y = \frac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

Solution to part a: Region R represents the area bounded by the curve and the horizontal line.

We first graph both equations. This problem is from the calculator section, so feel free to use your calculator to graph both and to see which one lies below the other.

On top of this, it's obvious that the intersection points $x = -3$ and $x = 3$. You can find this by simply setting the two equations equal to each other or by using the calculator.

Clearly between -3 and 3, $\frac{20}{1+x^2}$ lies above the other function. Let's say that $f(x) = \frac{20}{1+x^2}$ and $g(x) = 2$.

It's best to make infinite vertical strips in this problem since our equations are in terms of x. Making an infinite number of vertical strips with width dx will allow us to integrate our expression easily.

Each vertical strip has a height of $f(x) - g(x)$. The width of each stripe is dx.

We can set up our integral for the area as

$$\int_{-3}^{3} [f(x) - g(x)]dx = \int_{-3}^{3} [\frac{20}{1+x^2} - 2]dx$$

Since a calculator is allowed for this problem, we can enter this integral into the calculator and get an answer of 37.961

Part b has been skipped since it covers a topic that hasn't been introduced yet but will be soon.
Solution to part c: Since our cross sections are perpendicular to the x-axis, we want our equations to be in terms of x. Lucky for us, both $f(x)$ and $g(x)$ already are.

The diameter of each cross section is simply $f(x) - g(x)$. This means that the area of a semicircle is $\frac{\pi[f(x)-g(x)]^2}{8}$. Each cross section will also have a thickness of dx. We also know that the bounds are between $x = -3$ and $x = 3$.

We can write our integral for the volume as $\int_{-3}^{3} \frac{\pi[f(x)-g(x)]^2}{8}$. We can plug in our functions for $f(x)$ and $g(x)$ into the integral to get

$$\int_{-3}^{3} \frac{\pi[\frac{20}{1+x^2} - 2]^2}{8}$$

Since a calculator is allowed in this problem, we can enter this expression into the calculator and find that the volume is 174.268

§8.9 Volume with Disc Method: Revolving Around the x- or y-Axis

Now, we will be revolving one function around either the x or y axis. The goal will be to find the volume of the entire region that this function covers.

To solve such problems, it's best to visualize a single strip with respect to one of the axes.

Note

If we are asked to revolve around the x-axis, then we should follow the few steps that will be listed.

1. Make sure that everything is in terms of x so we can integrate it easily when we use dx.

2. If we revolve around the x-axis, then visualize a vertical strip of height $y = f(x)$. This vertical strip will be revolved around the x-axis. This will give us a circle with a tiny thickness of dx. The vertical strips height is $f(x)$ which is also the radius of this circle created by revolving around the x-axis.

This means that the area of a disk be written as $\pi[f(x)]^2$. The volume can be written as $\int_a^b \pi[f(x)]^2 dx$. This represents the volume of $f(x)$ revolved around the $x - axis$ between the points $x = a$ and $x = b$.

Note

If we are asked to revolve around the y-axis, then we should follow the few steps that will be listed.

1. Make sure that everything is in terms of y so we can integrate it easily when we use dy.

2. If we revolve around the y-axis, then visualize a horizontal strip of length $x = f(y)$. This horizontal strip will be revolved around the y-axis. This will give us a circle with a tiny thickness of dy. The vertical strips length is $f(y)$ which is also the radius of this circle created by revolving around the y-axis.

This means that the area can be written as $\pi[f(x)]^2$. The volume can be written as $\int_a^b \pi[f(y)]^2 dy$. This represents the volume of $f(y)$ revolved around the y-axis between the points $y = a$ and $y = b$.

Problem — Revolve $f(x) = \sqrt{x}$ around the x-axis when the bounds are $x = 2$ and $x = 6$. Find the volume.

Solution: Since we revolve around the x-axis, we will have an infinite number of vertical strips. Each vertical strip will be revolved around the x-axis.

The height of each vertical strip will simply be the distance between the function and the x-axis. Clearly, the length of each strip is just $f(x)$. This represents the radius of the disk that each vertical strip creates after it is rotated around the x-axis.

Since the radius is $f(x)$, the area of the region/disk that each vertical strip creates around the x-axis is $\pi[f(x)]^2$.

The thickness for each circle is just dx. We also know that our bounds are $x = 2$ and $x = 6$.

This means that the volume can be written as $\int_2^6 \pi[f(x)]^2 dx$

We plug in $f(x) = \sqrt{x}$ into the integral to get

$$\int_2^6 [\pi x] dx$$

The integral evaluates to $\frac{\pi x^2}{2}|_2^6$. This means the total volume is $\frac{\pi 6^2}{2} - \frac{\pi 2^2}{2} = 16\pi$

§8.10 Volume with Disc Method: Revolving Around Other Axes

Instead of revolving a function around the x or y-axis, now we will revolve it around $x = a$ or $y = b$ (a and b are both constants). They will be different axes that will be used.

We will again make an infinite number of circular discs to find the volume. The first step will be to find the radius. While in section 8.9 the radius was simply just the value of the function at that point, now we will have to subtract/add a value to it depending on our new axis.

If we rotate around $x = a$, then our entire integral should all be written in terms of x. Similarly, if we rotate around $y = b$, then the entire integral should be written in terms of y. The reason is that we want the variables used in the integral to match the variable we integrate with respect to. For example, if the area of each arbitrary disk is written in terms of x, then we should integrate with respect to x.

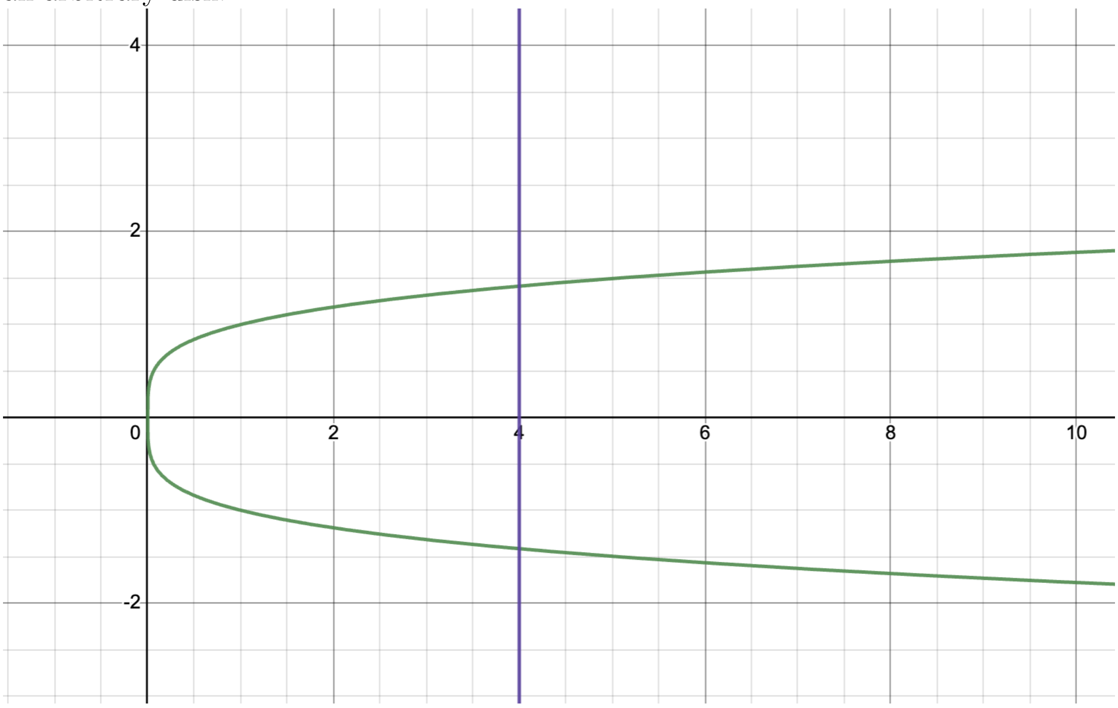
The goal of this method is the same as section 8.9. We will still find the radius of an arbitrary disk and write an expression for the area. Then, using either dx or dy, we will write an integral and integrate.

Problem — If $f(x) = \sqrt[4]{x}$ and the bounds are $y = -3$ and $y = 3$, find the volume of the function rotated around $x = 4$.

Solution: Since we are rotating around an axis that is parallel to the y-axis ($x = 4$), we should make sure that all functions are written in terms of y.

Clearly, $y = f(x) = \sqrt[4]{x}$ is written in terms of x. We can raise both sides to the power of 4 to get $x = y^4$. Now it is written in terms of y.

Now, we should find the length of a horizontal strip since that represents the radius of an arbitrary disk.



The graph makes it obvious that the length of a horizontal strip is simply $4 - y^4$ (the reason regarding why y^4 is subtracting instead of 4 is because 4 is to the right of y^4.

Now since the length of each horizontal strip is $4 - y^4$, each disk that it creates after rotation will have an area of $\pi[4 - y^4]^2$.

The thickness of each disk will be dy. We also already know that our bounds are $y = -3$ and $y = 3$.

We can write an integral expression for the volume as

$$\int_{-3}^{3} \pi[4 - y^4]^2 dy$$

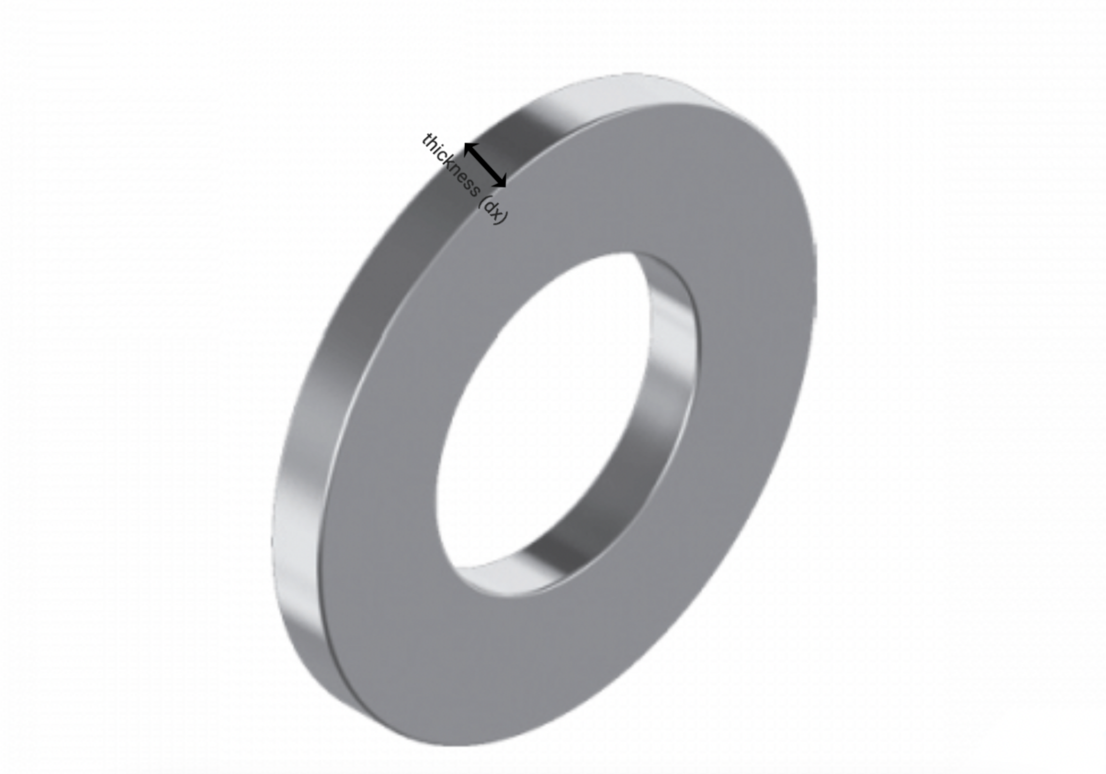
You can plug this integral into a calculator to get a volume of 3692.4

§8.11 Volume with Washer Method: Revolving Around the x- or y-Axis

Finding the volume using the washer method is trickier than the disc method.
The washer method is necessary when there are two functions in the problem. We will often rotate the area between the two functions and find the volume.

In simple words, the washer method finding the area between two disks, and integrating after to find the volume.

We will often have a function $f(x)$ above $g(x)$. Then, when we rotate this around the x axis, each cross section will have a hole inside a circle. A circle of smaller radius (from $g(x)$) will be cut out from a larger circle created by $f(x)$.



The above image is how the cross section looks like when we use the washer method. There will be a circle cut out inside of a larger circle. The larger circle will be created by the function that is larger/above the other.

For example, if $f(x)$ and $g(x)$ are rotated around the x-axis and $f(x)$ lies above $g(x)$, then $y = f(x)$ will represent the larger radius while $y = g(x)$ will represent the shorter one.

Note

How to apply the Washer Method when Revolving around the x-axis
Make sure to first write all of the functions in terms of x since we can integrate that with respect to x.
If we have two functions $f(x)$ and $g(x)$ that we rotate, make sure to graph both on an xy plane. This will allow us to know what function lies above the other.

We can set both functions equal to each other to find the integration bounds if they aren't given.

Now, we should imagine an infinite number of vertical strips. There will be one large strip that will be $y = f(x)$ units long. The smaller strip will be $y = g(x)$ units long. When we rotate these vertical strips around the x-axis, the area of this cross section that we want to find is the area of the entire disk of radius $f(x)$ minus the area of the disk with radius $g(x)$.

The area of the inner circle with radius $g(x)$ is simply $\pi[g(x)]^2$ while the radius of the outer circle with radius $f(x)$ is $\pi[f(x)]^2$. We can subtract the area of the inner circle from the outer one to get an area of $\pi[f^2(x) - g^2(x)]$.

Each cross section of this washer has thickness dx. This means the volume of each washer is $\pi[f^2(x) - g^2(x)]dx$.
We can integrate this for our bounds (assuming that they are a, b) to get a volume of

$$\int_a^b \pi[f^2(x) - g^2(x)]dx$$

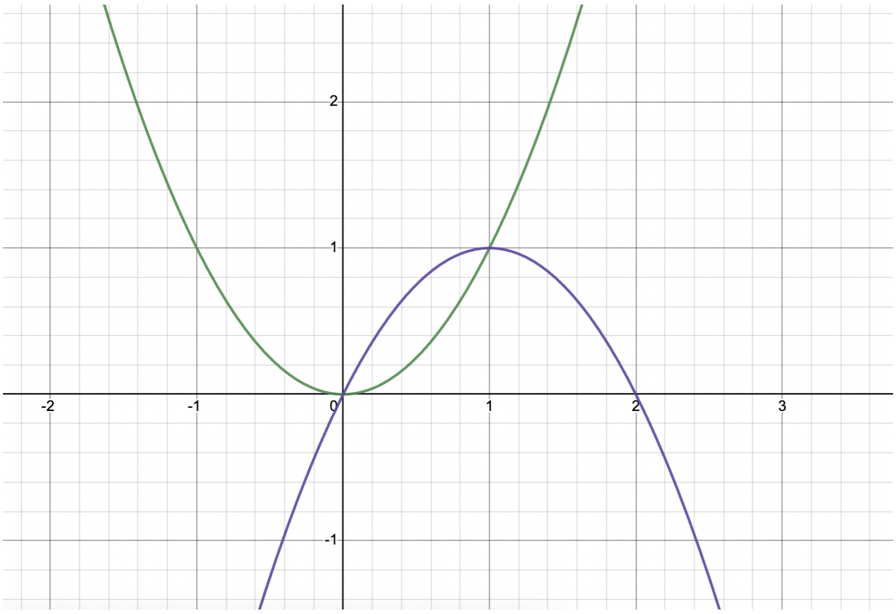
We must take similar steps if we revolve two functions around the y-axis and want to find the volume. We will wrote both equations in the form $x = f(y)$ and $x = g(y)$ since we want them to be in terms of y.

We also need to graph both functions to see which one lies 'above' the other.
Then, using our bounds (which can either be given or found by equating the functions), we can set up an integral. We know that each cross section will simple be a washer. Let's assume that the larger circle has a radius $f(y)$ while the smaller one has radius $g(y)$. Each washer cross section will have a thickness of dy.

We can set up an integral for the volume as $\int_a^b \pi[f^2(y) - g^2(y)]dy$. a and b are the bounds.
Finding the bounds might be confusing right now, but it will be cleared up with a few example in this unit.

Problem — Revolve the area of the region bounded by $f(x) = x^2$ and $g(x) = 2x - x^2$ around the x-axis.

Solution: We should first graph both functions to see which one lies above the other.



The green curves represents $y = x^2$ while the purple one represents $2x - x^2$. Clearly $g(x)$ lies further than $f(x)$ from the x-axis.

We can imagine making infinite vertical strips and rotating them around the x-axis. The larger strip will have a length of $g(x) = 2x - x^2$. Its area when revolved around the x-axis will be $\pi[g(x)]^2$.

The smaller strip will have a length of $f(x)$ and it will make our inner circle. Revolving it around the x-axis will give us an area of $\pi[f(x)]^2$.

We can subtract the smaller area from the larger one to get $\pi[g^2(x) - f^2(x)]$ which represents the area of one cross section. We will have an infinite amount of washer cross sections like this one. Each one will have a thickness of dx.

This means the volume of one cross section can be written as $\pi[g^2(x) - f^2(x)]dx$.

Now, we should find the bounds for integration. Clearly, we need to equate $f(x)$ and $g(x)$ to find the intersection points, which will give us our integration bounds.

Solving $x^2 = 2x - x^2$ gives that x can be 0 or 1. These are our bounds.

The integral for the volume is $\int_1^2 \pi[g^2(x) - f^2(x)]dx$.

We can plug in $f(x) = x^2$ and $g(x) = 2x - x^2$ into the above expression to get

$$\int_1^2 \pi[(2x - x^2)^2 - (x^2)^2]dx$$

This integral simplifies to $\int_1^2 \pi[4x^2 - 4x^3]dx$ which evaluates to a volume of $\frac{\pi}{3}$.

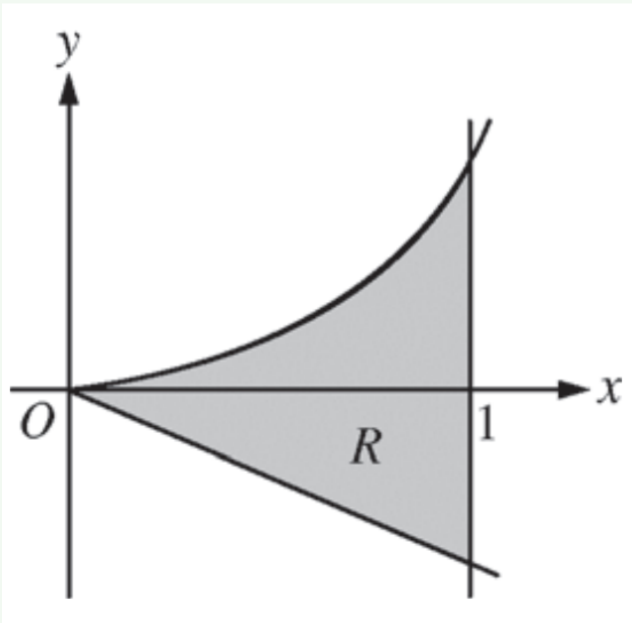
§8.12 Volume with Washer Method: Revolving Around Other Axes

Rotating around an axis that is not the x or y axis using the washer method is similar to the previous section. Instead of figuring our whether our function lies further from the x/y axis, we must figure out whether or not it lies further from the new axis.

This can be done by graphing both the new axis along with the given functions. The one that lies further will creates the outer circles while the one that lies closer to the axis will create the inner circles for each cross section after rotation.

The best way to learn this topic is to try a problem. The method of solving will be extremely similar to section 8.11 with a few tiny changes.

Problem — 2014 AP Calculus BC FRQ



Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do no evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

Solution to part a: This part is a review of earlier topics from this unit when we found area between two curves.

Since both equations are in terms of x, we should integrate with dx. This means we should imagine an infinite number of vertical strips. Each vertical strip will have length $xe^{x^2} + 2x$.

Each vertical strip will have width dx. We can integrate this with bounds $x = 0$ and $x = 1$ to get the integral $\int_0^1 [xe^{x^2} + 2x]dx$

$$\int_0^1 [xe^{x^2} + 2x]dx = \int_0^1 [xe^{x^2}]dx + \int_0^1 [2x]dx$$

We can separately find each integral and add both up.

$\int_0^1 [xe^{x^2}]dx$ can be found using u-substitution. We can plug in $u = x^2$ and integrate the equation using that to get that $\int_0^1 [xe^{x^2}]dx = \frac{e}{2} - \frac{1}{2}$.

$\int_0^1 [2x]dx$ can be found by using the anti-derivative of the power rule. We simply get that $\int_0^1 [2x]dx = x^2|_0^1 = 1$.

We can add up both integrals to get that

$$\int_0^1 [xe^{x^2} + 2x]dx = \int_0^1 [xe^{x^2}]dx + \int_0^1 [2x]dx = \frac{e}{2} - \frac{1}{2} + 1 = \frac{e}{2} = \frac{1}{2}$$

Solution to part b: To solve this part, we should notice that $y = -2$ is a line parallel to the x-axis. This means that all our functions should be in terms of x which already is the case.

Clearly, $y = xe^{x^2}$ lies further from $y = -2$ than $y = -2x$.
Let's say that $f(x) = xe^{x^2}$ and $g(x) = -2x$.

If we make an infinite number of vertical strips to rotate around $y = -2$, then the graph of $f(x)$ will be further than $g(x)$ relative to $y = -2$. Many people now jump to saying that the outer radius is $f(x)$ and the inner radius of each cross section is $g(x)$. However, this is NOT true.

The reason is that the axis we revolve is no longer the x-axis. It is $y = -2$. $f(x)$ lies 2 extra units further from $y = -2$ when compared to the x-axis. This means that $f(x) + 2$ will be the outer radius when we revolve the longer vertical strips. Similarly, $g(x) + 2$ will be the inner radius when we revolve the shorter vertical strips.

Clearly, the area of each washer cross section can be written as $\pi[(f(x)+2)^2 - (g(x)+2)^2]$. We multiply this to dx which is the thickness to find a volume of $\pi[(f(x) + 2)^2 - (g(x) + 2)^2]dx$.

We can integrate this by using the bounds from $x = 0$ and $x = 1$.

$$\text{We set up the integral } \int_0^1 \pi[(f(x) + 2)^2 - (g(x) + 2)^2]dx$$

We can substitute our functions $f(x)$ and $g(x)$ back into this to simplify the integral to

$$\int_0^1 \pi[(xe^{x^2} + 2)^2 - (-2x + 2)^2]dx$$

We can further simplify this by bringing our constant π out to get

$$\pi \int_0^1 [(xe^{x^2} + 2)^2 - (-2x + 2)^2]dx$$

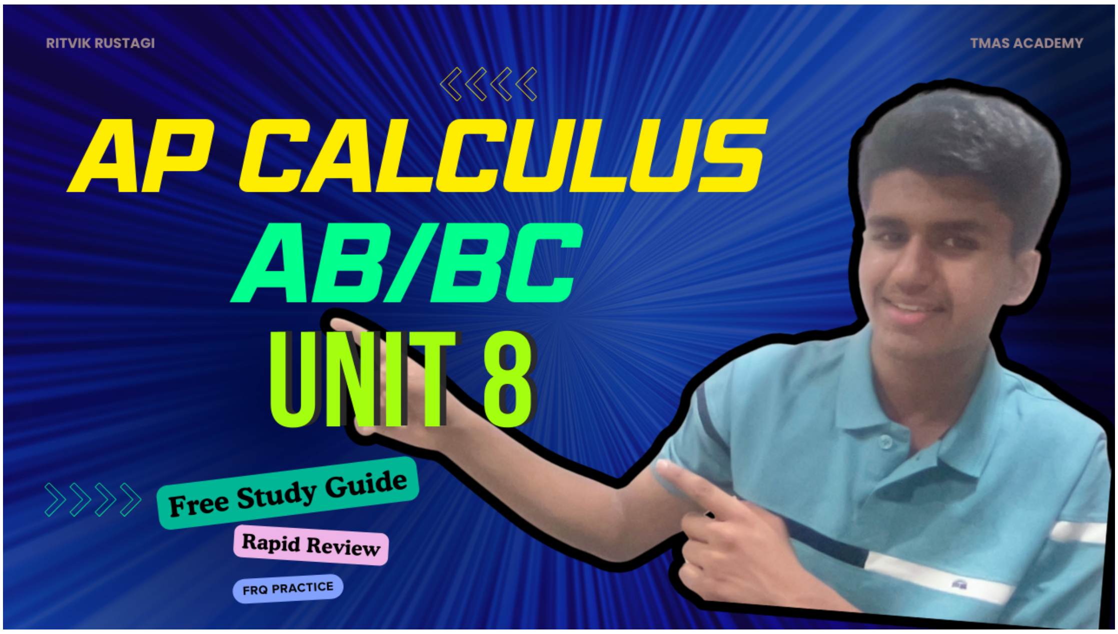
Unit 8 Practice Problems

Problem — Let R be the region in the first quadrant enclosed by the circle $x^2 + y^2 = 9$, the x-axis, $x = 0$, and $x = 3$.

(a) Find the volume of the solid generated when R is revolved about the x-axis.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the line $y = -2$.

Solution: Video Solution

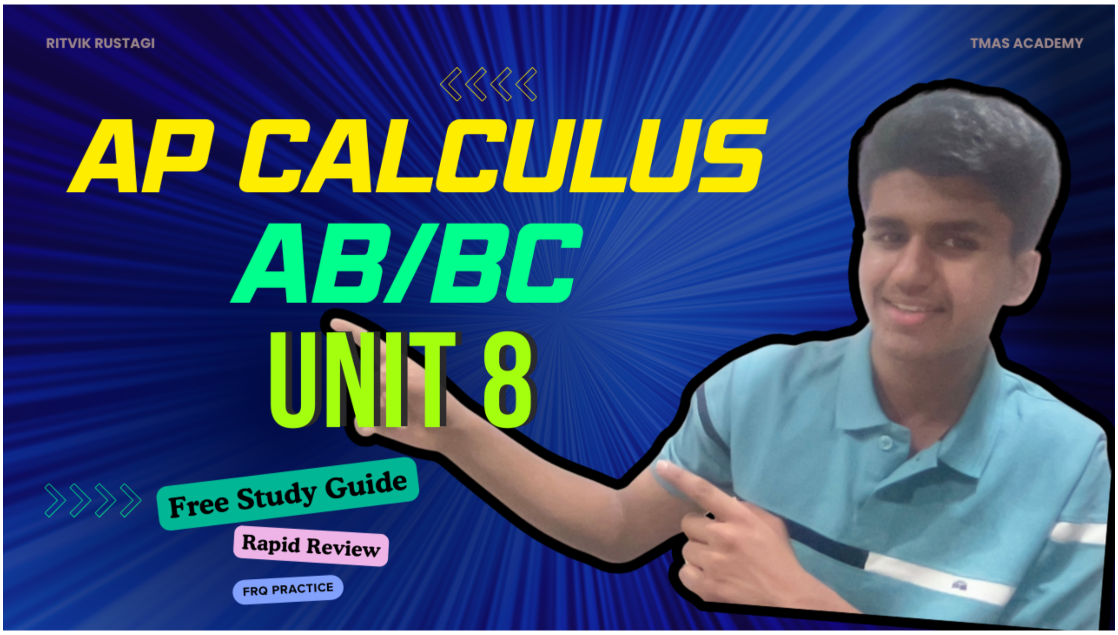


Problem — 1972 AP Calculus AB FRQ
The shaded region R is bounded by the graphs of $xy = 1$, $x = 1$, $x = 2$, and $y = 0$.

(a) Find the volume of the solid figure generated by revolving the region R about the x-axis.

(b) Find the volume of the solid figure generated by revolving the region R about the line $y = -2$.

Solution: Video Solution



Problem — 2011 AP Calculus AB FRQ
Let R be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

(b) Find the area of R.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

Solution to part a: To find the equation of the line tangent to the gaph of f at $x = \frac{1}{2}$, we will first find $f'(\frac{1}{2})$ since it represents the slope of the tangent line.

We know from the power rule that $f'(x) = 24x^2$. We can now plug in $\frac{1}{2}$ to find that $f'(\frac{1}{2}) = 24(\frac{1}{2})^2 = 6$

Thus, we can now plug in $m = 6$ (representing the slope) into the line $y = mx + b$ (equation of tangent line).

Doing so gives $y = 6x + b$. We know that this line passes through the point at $x = \frac{1}{2}$.

We must find the y-value of that point first. Thus, we plug in $x = \frac{1}{2}$ into $f(x)$ to get $f(\frac{1}{2}) = 8(\frac{1}{2})^3 = 1$

We now know that the tangent line passes through the point $(\frac{1}{2}, 1)$.
Plugging that point in gives $1 = 6 \cdot \frac{1}{2} + b$ which means that $b = -2$
This means the equation of the tangent line is $y = 6x - 2$

Solution to part b: The area of region R can be found by creating an infinite number of thin vertical strips. The height of each strip will be the difference of the two graphs. The width will be dx.

To find the difference in the two graphs, we must first figure out what function lies above the other. Clearly, $\sin(\pi x)$ lies above. Thus, the difference is $g(x) - f(x)$. We also know that the width is dx. We must integrate from $x = 0$ to $x = \frac{1}{2}$.

$$\text{Thus, the area is } \int_0^{1/2} [g(x) - f(x)]dx$$

We can substitute our functions to get $\int_0^{1/2}[\sin(\pi x) - 8x^3]dx$

$$\text{Integrating gives } -\frac{\cos(\pi x)}{\pi} - 2x^4|_0^{1/2}$$

$$\text{This means that the area is } -\frac{1}{\pi}\cos(\frac{\pi}{2}) - 2(\frac{1}{2})^4 - (-\frac{\cos(0)}{\pi} - 0) = \frac{1}{\pi} - \frac{1}{8}$$

Solution to part c: Clearly, after revolving region R about $y = 1$, we will have an infinite number of washers with a thickness of dx.
We should immediately be thinking about our washer methods and what our outer/inner radius will be.

The outer radius will be created by the graph that lies furthest from the axis of rotation for the region that we rotate. Clearly, $f(x) = 8x^3$ lies furthest from $y = 1$.
Thus, the outer radius is $1 - 8x^3$.

Similarly, the inner radius will be created by $g(x) = \sin(\pi x)$. The inner radius is $1 - \sin(\pi x)$
We know that the area of each washer will be the area of the larger circle minus the smaller one. We must integrate with dx (which represents the thickness of each washer.

$$\text{Thus, the volume is } \pi \int_0^{1/2} [(1 - 8x^3)^2 - (1 - \sin(\pi x))^2]dx$$

Problem — 2006 AP Calculus AB FRQ
Let R be the shaded region bounded by the graph $y = \ln x$ and the line $y = x - 2$, as shown above.
(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

Solution to part a: To find the area of R, we must first find the intersection points between the two graphs.
We can solve $\ln(x) = x - 2$ using a calculator to find that the two solutions occur when $x = 0.1586$ and $x = 3.1462$

The vertical distance between the two graphs can be used to find the area. Since $y = \ln x$ lies above $x - 2$ for region R, the vertical distance between the two graphs is

$\ln(x) - (x - 2) = \ln(x) - x + 2$

We know that there will be an infinite number of thin vertical strips that make the area. Each will have a width of dx.

Thus, the area is $\int_{0.1586}^{3.1462} (\ln(x) - x + 2) dx = 1.95$

Solution to part b: Each cross section will be a washer. We must find the area for a washer in general. For this, we need to find the outer radius and the inner radius for each washer.

The outer radius will be formed by the graph that lies furthest from the axis of rotation. Similarly, the inner radius will be formed by the graph that lies closest to the axis of rotation.

Clearly, for region R, $y = x - 2$ lies closer to $y = -3$ than $y = \ln x$.

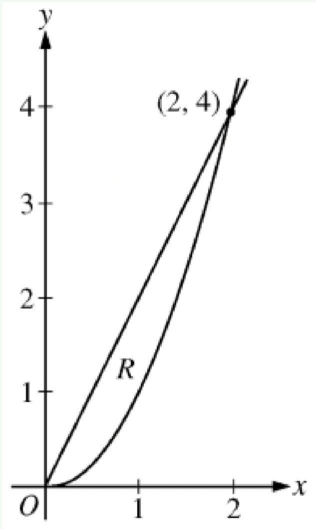
This means that the inner radius is $x - 2 - (-3) = x - 2 + 3 = x + 1$.

The outer radius is $\ln(x) - (-3) = \ln(x) + 3$

The area of each washer will be the area of the outer circle minus the area of the inner circle. Each washer will have a thickness dx.

Thus, the volume can be written as $\pi \int_{0.1586}^{3.1462} [(\ln(x) + 3)^2 - (x + 1)^2]$. We can use a calculator to find that the volume is 34.198

Problem — 2009 AP Calculus AB FRQ



Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin(\frac{\pi}{2}x)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

Solution to part a: We can find the area of R by subtracting both $y = 2x$ and $y = x^2$. That will give us an expression for the height for the infinite number of thin vertical strips. The width of each strip will be dx.

Clearly, $y = 2x$ lies above $y = x^2$. Thus, the height of the infinite amount of rectangles can be represented as $2x - x^2$.

$$\text{The area of R is } \int_0^2 (2x - x^2)dx = x^2 - \frac{x^3}{3}|_0^2$$
$$= 2^2 - \frac{2^3}{3} = \frac{4}{3}$$

Solution to part b: We know that each cross section will have a thickness of dx. We already know the area of each cross section to be $A(x) = \sin(\frac{\pi}{2}x)$.
Multiply the expression for the area by the thickness of dx will give the volume for each washer. We can integrate that expression to find the total volume.

Thus, the volume can be represented as

$$\int_0^2 A(x)dx = \int_0^2 \sin(\frac{\pi}{2}x)dx$$

$$= -\frac{2}{\pi}\cos(\frac{\pi}{2}x)|_0^2 = \frac{4}{\pi}$$

Solution to part c: Since the cross sections are now perpendicular to the y-axis, the thickness of each cross section will no longer be written as dx. It will be dy.

We know that the area of each cross section can be found by first finding the distance between the graphs in terms of y. The reason is that we are integrating with respect to y, so the area of each cross section should be in terms of y.

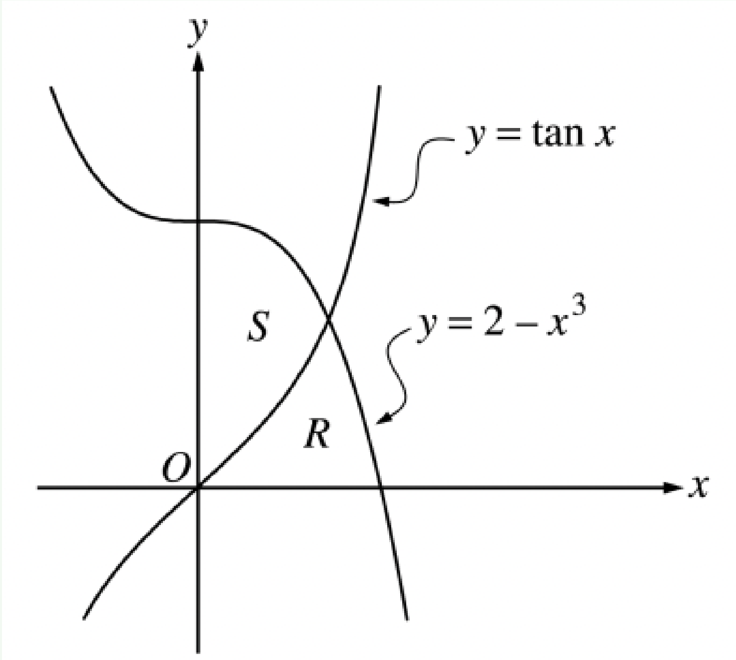
The two graphs are $x = \frac{y}{2}$ and $x = \sqrt{y}$.

We can graph both functions to see that $x = \sqrt{y}$ lies further from the y-axis than $x = \frac{y}{2}$. Thus, the distance between both is $\sqrt{y} - \frac{y}{2}$. We can square this to find the area of a square cross section since the side lengths of a square the same.

Thus, the area of each cross section can be written as $(\sqrt{y} - \frac{y}{2})^2$.

$$\text{The volume can be written as } \int_0^4 (\sqrt{y} - \frac{y}{2})^2 dy$$

Problem — 2001 AP Calculus AB FRQ



Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

(a) Find the area of R.
(b) Find the area of S.
(c) Find the volume of the solid generated when S is revolved about the x-axis.

Solution to part a: We can find the area of region R by splitting it into two separate parts. We can draw a line from the intersection point of the two graphs to the x-axis to split it up.

Now, we must find the intersection points so we can find the integration bounds for each part of region R.

We know that $2 - x^3 = \tan x$. We use a calculator to find that $x = 0.902$ and $y = 1.266$

We can integrate $y = \tan x$ from $x = 0$ to $x = 0.902$. Then, we can also integrate $y = 2 - x^3$ from $x = 0.902$ to $x = 1.266$ and sum up the areas of both.

$$\text{Area of Region B: } \int_0^{0.902} \tan x \, dx + \int_{0.902}^{1.266} (2 - x^3)dx$$

We can plug this into a calculator to find that the area is 0.729

Solution to part b: There are many ways to find the area of S.
We will again make an infinite number of thin vertical strips in region S. Each will have a width of dx. This means that the height should be in terms of x so we can easily integrate.

$y = 2 - x^3$ is the function on the top since it is greater than $y = \tan x$ for region S.
This means that the area of S is $\int_0^{0.902}(2 - x^3 - \tan x)dx = 1.16$

Solution to part c: When we revolve region S around the x-axis, each washer will have a thickness of dx. The vertical strips from $y = 2 - x^3$ will create the outer disk since this curve lies above $\tan x$ in our region.

The inner disk will be made by $y = \tan x$.

The area of each washer will be the area of the disk with a larger radius minus the area of the disk with a smaller radius.

This means that our integral can be written as

$$\int_0^{0.902}[\pi(2 - x^3)^2 - \tan^2 x]dx = 8.331$$

Problem — 2002 AP Calculus AB FRQ
Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.
(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

Solution to part a: We can integrate for the area enclosed by the graphs with dx. dx will represent the width of the infinite number of thin vertical strips that will be made.

The height of each strip will be the difference in the y-value for both functions. Now, we must figure out which function lies above the other. From graphing, we can see that e^x lies above $\ln x$ when x is between $\frac{1}{2}$ and 1.

Thus, the area enclosed by the graphs can be written as

$$\int_{1/2}^{1}(e^x - \ln x)dx = 1.222$$

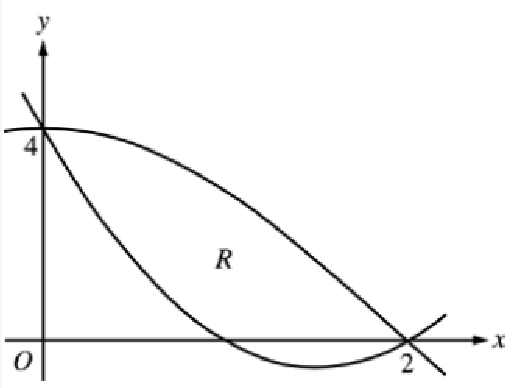
Solution to part b: $y = 4$ will be the axis that we revolve around. We must integrate with dx which represents the thickness of each washer.

We must figure out which function lies furthest from $y = 4$. In this problem, e^x will also be closer to $y = 4$ than $\ln x$ when x is between $\frac{1}{2}$ and 1. This means that the inner radius will be created by $y = e^x$ while the outer radius will be created by $\ln x$.

The expression for the inner radius will $4 - e^x$ while the outer radius will be $4 - \ln x$
The volume can be written as

$$\pi\int_{1/2}^{1}[(4 - \ln x)^2 - (4 - e^x)^2]dx = 23.609$$

Problem — 2013 AP Calculus AB FRQ



Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos(\frac{1}{4}\pi x)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

Solution to part a: Since both functions f and g are written in terms of x, it makes sense to integrate with respect to x.

We can do this by making an infinite number of thin vertical strips in region R. The height of each strip will be the difference between the two functions. The width of each rectangle will be dx.

To find the height, we must figure out which function lies above the other.
It's noticeable that $g(x)$ lies above $f(x)$. Thus, the height of each rectangle is $g(x) - f(x)$.

The area can be represented as $\int_0^2 [g(x) - f(x)]dx$.

$$4\int_0^2 [g(x) - f(x)]dx = \int_0^2 [4\cos(\frac{1}{4}\pi x) - (2x^2 - 6x + 4)]dx$$

$$= \int_0^2 [4\cos(\frac{1}{4}\pi x)]dx - \int_0^2 (2x^2 - 6x + 4)dx = \frac{16}{\pi}\sin(\frac{1}{4}\pi x)|_0^2 - (\frac{2x^3}{3} - 3x^2 + 4x|_0^2)$$

$$= \frac{16}{\pi}[\sin(\frac{\pi}{2}) - \sin(0)] - (\frac{16}{3} - 12 + 8) = \frac{16}{\pi} - \frac{4}{3}$$

Solution to part b: Clearly, this is a problem involving washers since we are rotating a region bounded by two graphs.
We must figure out which function creates the outer radius while which one creates the inner radius for each washer.

The function furthest from the axis of rotation will create the outer radius for each washer after being revolved.

Clearly, $f(x)$ lies furthest from $y = 4$. That means an expression for the outer radius is $4 - f(x)$ while for the inner radius it is $4 - g(x)$.

The area of each washer will be the area of the large circle minus the area of the small circle. Each washer will have a thickness of dx.
That means the volume can be represented as

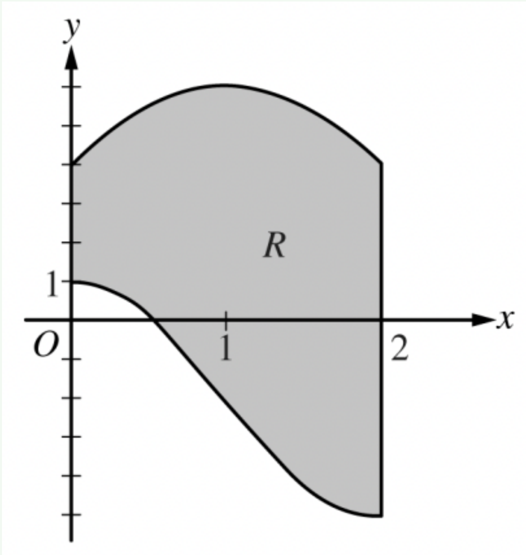
$$\pi \int_0^2 [(4 - f(x))^2 - (4 - g(x))^2] dx$$

$$= \pi \int_0^2 [(4 - (2x^2 - 6x + 4))^2 - (4 - 4\cos(\frac{1}{4}\pi x))^2] dx$$

Solution to part c: Since each cross section is taken perpendicular to the x-axis, we want the area of each cross section to be in terms of x. Then, we can integrate it with respect to dx (dx represents the thickness of each cross section)

The length of the base of each square will simply be $g(x) - f(x)$. Since a square has all sides that are equal, the area of this square will be $[g(x) - f(x)]^2$. We can now integrate this with dx.
The volume is $\int_0^2 [g(x) - f(x)]^2 dx = \int_0^2 [4\cos(\frac{1}{4}\pi x) - (2x^2 - 6x + 4)]^2 dx$

Problem — 2019 AP Calculus AB FRQ



Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos(\frac{\pi}{2}x)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

Solution to part a: The area of R can be found by taking an infinite amount of thin vertical strips. Each one will have a height that is a difference of the two functions. The width will simply be dx.

We must figure out which graph lies above the other. Clearly, $h(x)$ lies above $g(x)$ for region R. Thus, the height is $h(x) - g(x)$.

$$\text{The area of R: } \int_0^2 [h(x) - g(x)]dx$$

$$= \int_0^2 [6 - 2(x-1)^2 - (-2 + 3\cos(\frac{\pi}{2}x))]dx$$

$$= [6x - \frac{2}{3}(x-1)^3 - (-2x + \frac{6}{\pi}\sin(\frac{\pi}{2}x))]|_0^2$$

$$= ((12 - \frac{2}{3}) - (-4 + 0)) - ((0 + \frac{2}{3}) - 0) = \frac{44}{3}$$

Solution to part b: We already know the area of each cross section to be $A(x)$. Each cross section will have a thickness of dx.

Thus, the volume is $\int_0^2 A(x)dx = \int_0^2 \frac{1}{x+3}dx = \ln(x+3)|_0^2$

We can find that $\ln(x+3)|_0^2 = \ln 5 - \ln 3 = \ln \frac{5}{3}$ (this is the volume of our solid).

Solution to part c: Whenever we are rotating about an axis (and we make washers), we must find out which graph will revolve to create the outer radius while which one creates the inner radius for each washer.

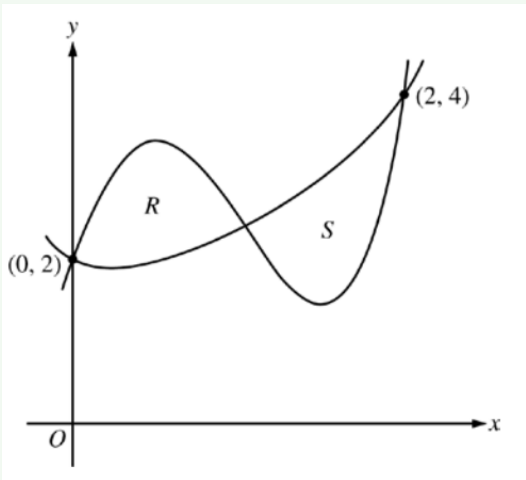
The outer radius will be created by the graph that lies furthest from the axis of rotation. In this case, the axis of rotation is $y = 6$.

Clearly, $g(x)$ lies furthest from the axis of rotation. Thus, the outer radius is simply $6 - g(x)$.
Similarly, the inner radius will be $6 - h(x)$.

The cross sectional area of each washer will be the area of the larger circle minus the area of the smaller circle. The thickness of each washer will be dx.
Thus, the volume can be represented by

$$\pi \int_0^2 [(6 - g(x))^2 - (6 - h(x))^2] dx$$

Problem — 2015 AP Calculus AB FRQ



Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2-2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

Solution to part a: We must separately find the areas of region R and S.

Before we do any integration, we will find the intersection points of $f(x)$ and $g(x)$. From the graph, we can tell that both of them intersect at $x = 0$ and $x = 2$.

However, there is another intersection point in the middle.

We can put the equation $1 + x + e^{x^2-2x} = x^4 - 6.5x^2 + 6x + 2$ into our calculator to find the other point.

The other intersection point is $(1.033, 2.401)$.

For region R, we can make an infinite amount of thin vertical strips, each with a width of dx. The height of each vertical strip will be the difference of the two functions.

To find the height, we must find out which graph lies above the other. For region R, $g(x)$ lies above $f(x)$. Thus, the height is $g(x) - f(x)$.

The area of region R is $\int_0^{1.033} [g(x) - f(x)]dx = 0.997$

Now, for region S, we will again make an infinite amount of thin vertical strips, each with a width of dx. The height of each vertical strip will be the difference of the two functions.

We must figure out which graph lies above the other. Even though for region R $g(x)$ was above $f(x)$, for region S $f(x)$ lies above $g(x)$.

This means that the height of each thin vertical strip in region S is $f(x) - g(x)$.

We can write the area as $\int_{1.033}^{2} [f(x) - g(x)]dx = 1.007$

We can now sum up both of the areas to find that the total area of both regions is $0.997 + 1.007 = 2.004$

Solution to part b: We must find the base length of the square as it will allow us to find the area.

It is simply the difference between the two functions which is $f(x) - g(x)$. The area of a square is simply the side length squared (since all sides are the same).

Thus, the area of each cross section will be $[f(x) - g(x)]^2$
We know that the volume of the 3D shape will be $\int_a^b A(x)dx$ where a and b are the integration bounds and $A(x)$ represents the area of a cross section in general.

Using the expression above, we write the volume as $\int_{1.033}^{2} [f(x) - g(x)]^2 dx = 1.283$

Solution to part c: Since $f(x)$ lies above $g(x)$ in region S, the vertical distance will be $f(x) - g(x)$.

Thus, $h(x) = f(x) - g(x)$

To find $h'(1.8)$, we first find the first derivative of our function $h(x)$.

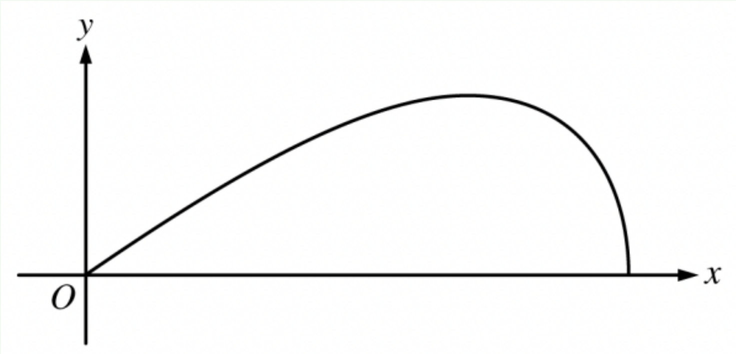
Doing so gives $h'(x) = f'(x) - g'(x)$

Plugging in $x = 1.8$ gives $h'(1.8) = f'(1.8) - g'(1.8)$

We can now find $f'(1.8)$ and $g'(1.8)$ through our calculators.

Doing so gives that $h'(1.8) = -3.812$

Problem — 2021 AP Calculus AB FRQ



A company designs spinning toys using the family of functions $y = cx\sqrt{4-x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4-x^2}$, $\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Solution to part a: To find the area in the first quadrant, we will make an infinite number of thin vertical strips. Each will have a width of dx.

We must first figure out our integration bounds. We already know that it starts at $x = 0$. We can find the other bound by writing the equation: $cx\sqrt{4-x^2} = 0$
(we write this equation since we know our function is integrated above the x-axis, so we figure out when y is 0 since that is when the function will hit the x-axis).

We can solve it to find that $x = 0, -2, 2$. The solution that we want is 2 since its part of the first quadrant.

We also know that the height of every thin vertical strip will be $cx\sqrt{4-x^2}$.

Thus, the area is $\int_0^2 [cx\sqrt{4-x^2}]dx$.

We can plug in $c = 6$ to get $\int_0^2 [6x\sqrt{4-x^2}]dx$

Now, to integrate this, we will use u-substitution.
We make the substitution $u = 4 - x^2$.

Now we can differentiate both sides to get $du = -2xdx$

We can plug in both of those to turn the integral to $\int_0^2 [-3\sqrt{u}]du$.
We must also change the integration bounds. We plug in $x = 0$ and $x = 2$ into $u = 4 - x^2$ to find that the new integration bounds are $u = 4$ to $u = 0$.

Thus, the new integral is $\int_4^0 [-3\sqrt{u}]$.

$$\text{Integrating gives } -2u^{3/2}|_4^0 = 0 - (-2(4)^{3/2}) = 16$$

Thus, we are able to find that the area is 16 square inches.

Solution to part b: The largest radius will occur when $y = cx\sqrt{4-x^2}$ is furthest from the x-axis. Thus, we are trying to find the relative extrema.

Relative extrema can occur at critical points. We will now find our critical points.
Critical points will exist when $\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0$ or is undefined. It will be 0 when $x = \pm\sqrt{2}$ and it will be undefined when $x = \pm 2$.

We can disregard the negative x values since they won't be part of the first quadrant. Thus, we only need to check $\sqrt{2}$ and 2.

However, the relative extrema has to occur when x is greater than 0 but less than 2 to satisfy the integration bounds for x.

Thus, we know that $x = \sqrt{2}$.
We also know that the largest radius (y-value) is 1.2

We can plug this information into the equation $y = cx\sqrt{4-x^2}$

Doing so gives $1.2 = c\sqrt{2} \cdot \sqrt{4 - (\sqrt{2})^2} = 2c$

Since $2c = 1.2$, we can find that $c = 0.6$

Solution to part c: First, we must realize that this is a disk problem. The reason is that we are only dealing with one graph of $y = cx\sqrt{4-x^2}$.
Since we revolve around the x-axis, each of the disks will have a thickness of dx.

The radius of each disk will be $y = cx\sqrt{4-x^2}$. That means the area of each disk will be $\pi(cx\sqrt{4-x^2})^2$

We can find that the volume for any spinning toy in general can be represented as

$$\pi \int_0^2 [cx\sqrt{4-x^2}]^2 dx = \pi \int_0^2 [c^2x^2(4-x^2)]dx$$

We can bring c^2 out of the integral since it's a constant. Doing so gives

$$\pi c^2 \int_0^2 [x^2(4-x^2)]dx$$

Now, we can expand the expression in the inside of the integral to get

$$\pi c^2 \int_0^2 [4x^2 - x^4]dx$$

The integral evaluates to

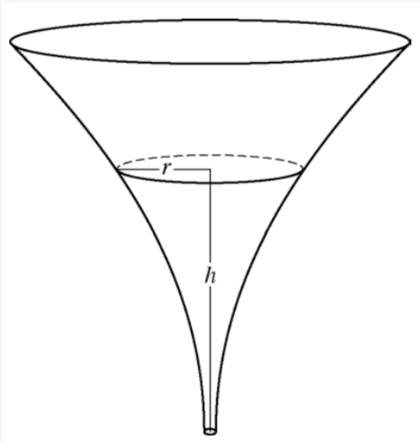
$$\pi c^2 (\frac{4x^3}{3} - \frac{x^5}{5}|_0^2) = \pi c^2 (\frac{4(2)^3}{3} - \frac{2^5}{5}) = \frac{64\pi c^2}{15}$$

We know that the volume of the spinning toy that we are working with is 2π. Thus, we can set the expression we found for volume to this value. This will allow us to find c. We can write the equation $\frac{64\pi c^2}{15} = 2\pi$

We can multiply both sides by $\frac{15}{64}$ to get $\pi c^2 = \frac{15\pi}{32}$

Now, we can divide π from both sides to get $c^2 = \frac{15}{32}$. Thus, square rooting both sides gives that $c = \sqrt{\frac{15}{32}}$.

Problem — 2016 AP Calculus BC FRQ



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$ where $0 \leq h \leq 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

Solution to part a: To find the average radius, we will find the sum of all the radii throughout the funnel. We are already given an expression for radius in terms of h.

If we integrate that, then we will be able to find the sum of all radii. Then, we can divide it by the length of the interval that we integrate for to find the average value. (This topic should remind you of the average value of a function which was covered previously).

$$\frac{1}{10-0}\int_0^{10} \frac{1}{20}(3+h^2)dh = \frac{1}{200}(3h + \frac{h^3}{3})|_0^{10}$$
$$= \frac{1}{200}(3 \cdot 10 + \frac{10^3}{3}) = \frac{1}{200} \cdot \frac{109}{60} \text{ in.}$$

Solution to part b: In our funnel, we have an infinite number of circular cross sections. To find the total volume, we can find the volume of all the circular cross sections and sum them up. The height of each circular cross section will be dh.

The area of each cross section will be πr^2. This means that the volume can be represented as

$$\int_0^{10} \pi r^2 \, dh = \pi \int_0^{10} \left[\frac{1}{20}(3+h^2)\right]^2 dh$$

$$= \frac{\pi}{400}\int_0^{10}[3+h^2]^2 dh = \frac{\pi}{400}\int_0^{10}[9+6h^2+h^4]dh$$

We can continue simplifying our integral to get

$$\frac{\pi}{400}\int_0^{10}[9+6h^2+h^4]dh = \frac{\pi}{400}[9h+2h^3+\frac{h^5}{5}]|_0^{10}$$

$$= \frac{\pi}{400}[9\cdot 10+2(10)^3+\frac{10^5}{5}] = \frac{2209\pi}{40}\text{in}^3$$

Solution to part c: In the part, we are dealing with multiple rates of changes. This should immediately cause us to think about related rates.

$$\text{We know that } \frac{dr}{dt} = \frac{dr}{dh}\cdot\frac{dh}{dt}$$

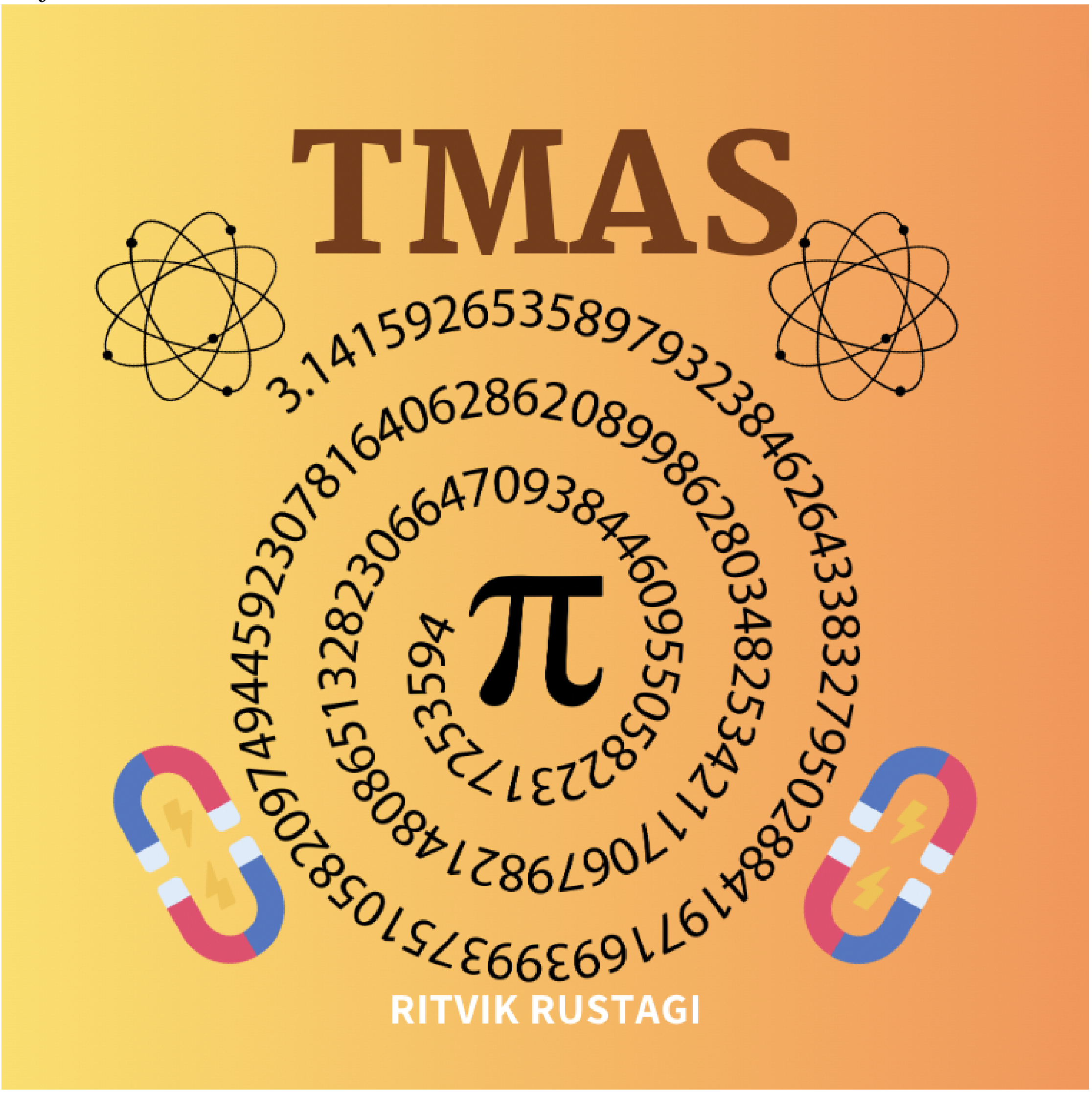
We are already given that $\frac{dr}{dt} = -\frac{1}{5}$. On top of that, we can find $\frac{dr}{dh}$ since we are given an expression for radius in terms of the height.
We know that $r = \frac{1}{20}(3+h^2)$. Thus, $\frac{dr}{dh} = \frac{h}{10}$.

We can plug all of this in to find that $-\frac{1}{5} = \frac{h}{10}\cdot\frac{dh}{dt}$.
We also know that the height is 3 at this time. We can plug in $h=3$ to get $-\frac{1}{5} = \frac{3}{10}\cdot\frac{dh}{dt}$.

Now we multiply both sides by $\frac{10}{3}$ to find that

$$\frac{dh}{dt} = -\frac{2}{3}\text{ in/sec}$$

Thank you for going through this book!
It is an honor for me to have contributed to your academical journey in some way!



Thanks,

Ritvik Rustagi