πMA∫ Academy

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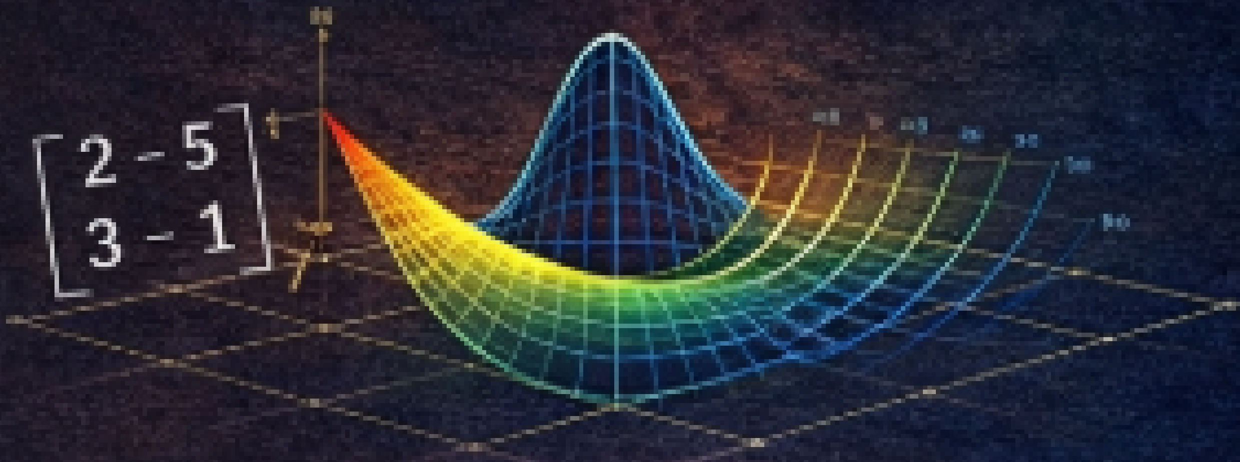
Differential Equations and Linear Algebra



2026

- 100+ Problems
- All Topics
- Detailed Solutions

Aditya Baisakh

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0.1 About TMAS Academy

The Math and Science (TMAS) Academy, formerly known as **Explore Math**, is a nonprofit organization which was started in 2021 to spread competition math resources to those who may not be able to afford existing ones which are usually overpriced. We believe no student should have to struggle merely due to a lack of resources, that a quality education is a right for everyone and should not be dependent on the size of their wallets. Later, it was expanded in March 2024 after the team started developing resources to assist students in preparation for AP STEM courses and exams. These included books from AP Calculus to AP Physics to AP Chemistry to AP Computer Science and more. Currently, the team is working on developing more specialized resources and expand the audience to include university students as well.

0.2 Opportunities For You To Contribute To TMAS Academy

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Donations: If you want to assist us in our monthly payments to run this organization, which includes website costs, Overleaf costs (the platform used to write these books), and filming/editing costs, then please consider donating! For those who are willing to contribute, we have listed some ways below. **Don't forget to write a message so we know who you are and can send you a thank you note!**

- You can donate through PayPal to the email: weexploremath@gmail.com
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Also, don't forget to join the Discord server to connect with other hardworking students preparing for AP exams and math competitions such as AMC 10/12 and AIME. `https://discord.gg/tmas-academy-1019082642794229870`

You can also follow all of our social media such as the LinkedIn page and the Instagram account that is run by the marketing team. Also, please join the mailing list to learn about all updates and our upcoming books and videos.

Finally, you can spread our efforts and initiative to anyone you know who may benefit from or support us, be it your classmates, teachers, or other nonprofit organizations focused on education.

0.3 About the Author: Aditya Baisakh

My name is Aditya Baisakh, and I am a freshman at Louisiana State University with a strong passion for mathematics and education. Although my interest in math was present early on, it spiked during high school as I became active in contests such as the AMC 10/12 and Mu Alpha Theta, where I developed a love for creative problem solving.

During my first semester at LSU, I took Differential Equations and Linear Algebra. This experience challenged me to think more deeply about mathematics and ultimately became one of the most rewarding courses of my first year. Through consistent practice and curiosity, I was fortunate to complete the course with an A+.

This book grew out of the notes I wrote while studying the material. My goal is to make these ideas clearer and more accessible for students who want to explore the beauty and power of mathematics, free of cost.

0.4 What if there is an error in the book?

There are possibilities for errors such as typos or incorrect solutions to problems. If that is the case, please click on this link and fill out the form to report the error:
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0.5 Any other questions or concerns?

If you have any questions about TMAS Acaemy and its programs, please contact Aditya Baisakh, the author and current CEO of TMAS Academy at the address
`adityabaisakh123@gmail.com`.

0.6 Acknowledgements

- As the CEO of such a wonderful student organization, I want to thank the entire TMAS Academy community for their unwavering support of the team's educational initiative and for being my inspiration to always put in my best effort.
- I would also like to thank the **Art of Problem Solving (AoPS)** for their supportive community of math and science enthusiasts. In addition, their LaTeX programming forums and tutorials were extremely helpful in writing this book.
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- I would also like to thank everyone who has supported the work I have done and encouraged me to continue.
- Finally, I want to thank my parents for their unwavering support of my ambitions and for everything else they have done.

1 First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

Much of mathematics is devoted to describing patterns and laws that arise in nature. For static relationships, algebraic equations are often sufficient. However, many of the most interesting phenomena in science involve *change*, and these are naturally described by equations involving derivatives. Such equations are called **differential equations**.

Recall from calculus that if a quantity $x = f(t)$ depends on an independent variable t, then its derivative

$$\frac{dx}{dt} = f'(t)$$

represents the instantaneous rate of change of x with respect to t. An equation that relates an unknown function to one or more of its derivatives is called a **differential equation**.

For example, the equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves the unknown function $x = x(t)$ together with its first derivative. Likewise,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function $y = y(x)$ together with its first and second derivatives.

A classical algebra problem asks us to determine the unknown *numbers* satisfying an equation such as

$$x^3 + 5x^2 - 11x + 37 = 0.$$

By contrast, when solving a differential equation, we seek the unknown *functions* $y = y(x)$ for which a relation such as

$$y'(x) = 2xy(x)$$

holds for all x in some interval. Equivalently, we seek all functions satisfying the differential equation

$$\frac{dy}{dx} = 2xy.$$

Example Let C be a constant and define

$$y(x) = Ce^{x^2}.$$

Then

$$\frac{dy}{dx} = C\frac{d}{dx}\left(e^{x^2}\right) = C\left(2xe^{x^2}\right) = 2x\left(Ce^{x^2}\right) = 2xy(x).$$

Thus every function of the form

$$y(x) = Ce^{x^2}$$

is a solution of the differential equation

$$\frac{dy}{dx} = 2xy.$$

This gives an infinite family of solutions, parameterized by the arbitrary constant C. Later, by using separation of variables, we will show that every solution of this differential equation is in fact of this form.

Differential Equations and Mathematical Modeling Many scientific laws can be expressed naturally in the language of differential equations. In the examples below, the independent variable will be time t, although in general other variables may also serve as independent variables.

Newton's Law of Cooling. This law states that the time rate of change of the temperature $T(t)$ of a body is proportional to the difference between the temperature of the body and the ambient temperature A of the surrounding medium. Hence

$$\frac{dT}{dt} = -k(T - A),$$

where $k > 0$ is a constant.

Notice that if $T > A$, then $\frac{dT}{dt} < 0$, so the body cools down. On the other hand, if $T < A$, then $\frac{dT}{dt} > 0$, so the body warms up. Thus the differential equation captures the expected physical behavior.

Torricelli's Law. Torricelli's law implies that the time rate of change of the volume $V(t)$ of water in a draining tank is proportional to the square root of the depth $y(t)$ of the water. Thus

$$\frac{dV}{dt} = -k\sqrt{y},$$

where $k > 0$ is a constant.

Suppose now that the tank is cylindrical with vertical sides and constant cross-sectional area A. Then

$$V = Ay, \qquad \frac{dV}{dt} = A\frac{dy}{dt}.$$

Substituting into the previous equation gives

$$A\frac{dy}{dt} = -k\sqrt{y},$$

or equivalently,

$$\frac{dy}{dt} = -h\sqrt{y},$$

where $h = \frac{k}{A}$ is a positive constant.

Population Growth. A simple model for population growth assumes that the time rate of change of a population $P(t)$ is proportional to the population itself. In that case,

$$\frac{dP}{dt} = kP,$$

where k is a constant.

We claim that every function of the form

$$P(t) = Ce^{kt}$$

is a solution. Indeed,

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t),$$

so the differential equation is satisfied for all t.

Even when the value of k is known, the equation $\frac{dP}{dt} = kP$ has infinitely many solutions, one for each choice of the constant C.

Now suppose that $P(t)$ is the population of a bacterial colony, that t is measured in hours, that

$$P(0) = 1000,$$

and that the population doubles after one hour, so that

$$P(1) = 2000.$$

Using the general solution $P(t) = Ce^{kt}$, the condition $P(0) = 1000$ gives

$$1000 = P(0) = Ce^0 = C,$$

so $C = 1000$. Next,

$$2000 = P(1) = 1000e^k,$$

hence

$$e^k = 2 \qquad \Longrightarrow \qquad k = \ln 2 \approx 0.693147.$$

Therefore the differential equation becomes

$$\frac{dP}{dt} = (\ln 2)P,$$

and the corresponding particular solution is

$$P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t.$$

This formula can now be used to predict future population values. For example, after one and a half hours, the population of the colony will be

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$

We now consider some practice problems.

Problem 1.1.1. *Verify that the function*

$$y(x) = 2x^{1/2} - x^{1/2}\ln x$$

satisfies the differential equation

$$4x^2y'' + y = 0$$

for all $x > 0$.

Solution. First compute the derivatives. Writing

$$y(x) = 2x^{1/2} - x^{1/2} \ln x,$$

we obtain

$$y'(x) = x^{-1/2} - \left(\frac{1}{2}x^{-1/2} \ln x + x^{-1/2}\right) = -\frac{1}{2}x^{-1/2} \ln x.$$

Differentiating once more,

$$y''(x) = -\frac{1}{2}\frac{d}{dx}\left(x^{-1/2} \ln x\right) = -\frac{1}{2}\left(-\frac{1}{2}x^{-3/2} \ln x + x^{-3/2}\right) = \frac{1}{4}x^{-3/2} \ln x - \frac{1}{2}x^{-3/2}.$$

Substituting into the differential equation gives

$$\begin{aligned} 4x^2y'' + y &= 4x^2\left(\frac{1}{4}x^{-3/2} \ln x - \frac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2} \ln x \\ &= x^{1/2} \ln x - 2x^{1/2} + 2x^{1/2} - x^{1/2} \ln x \\ &= 0. \end{aligned}$$

Hence $y(x)$ satisfies the differential equation for all $x > 0$.

It is important to note that writing down a differential equation does not guarantee the existence of a real-valued solution. For instance,

$$(y')^4 + y^2 = -1$$

has no real-valued solution, since the left-hand side is always nonnegative. On the other hand,

$$(y')^4 + y^2 = 0$$

has exactly one real-valued solution, namely

$$y(x) \equiv 0.$$

Thus some differential equations have no solution, some have exactly one, and some have infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. For example,

$$y^{(4)} + x^3y^{(3)} - 3x^5y = \sin x$$

is a fourth-order differential equation.

More generally, the most general form of an n**th-order differential equation** for an unknown function $y = y(x)$ is

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0,$$

where F is a given real-valued function of $n + 2$ variables.

As another example, let

$$y(x) = A\cos 3x + B\sin 3x,$$

where A and B are constants. Then

$$y'(x) = -3A\sin 3x + 3B\cos 3x$$

and

$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x).$$

Therefore every function of the form

$$y(x) = A\cos 3x + B\sin 3x$$

satisfies the differential equation

$$y'' + 9y = 0.$$

This is a two-parameter family of solutions.

So far, all of our examples have involved **ordinary differential equations**, meaning that the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, then the corresponding equation is called a **partial differential equation**.

For example, the temperature $u = u(x,t)$ of a long, thin, homogeneous rod at position x and time t satisfies the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where $k > 0$ is the thermal diffusivity.

In this chapter, however, we will focus primarily on first-order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x,y).$$

A particularly important problem is the **initial value problem**, which consists of a differential equation together with an **initial condition**

$$y(x_0) = y_0.$$

Thus an initial value problem has the form

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0.$$

Here x_0 need not be zero. Solving such a problem means finding a differentiable function $y(x)$ that satisfies both the differential equation and the initial condition on some interval containing x_0.

Problem 1.1.2. *Given that*

$$y(x) = \frac{1}{C - x}$$

is a solution of the differential equation $\frac{dy}{dx} = y^2$*, solve the initial value problem*

$$\frac{dy}{dx} = y^2, \qquad y(1) = 2.$$

Solution. We seek a value of C such that

$$y(x) = \frac{1}{C - x}$$

satisfies the initial condition $y(1) = 2$. Substituting $x = 1$ gives

$$2 = y(1) = \frac{1}{C - 1}.$$

Hence

$$C - 1 = \frac{1}{2}, \qquad C = \frac{3}{2}.$$

Therefore

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

So the solution of the initial value problem is

$$\boxed{y(x) = \frac{2}{3 - 2x}}.$$

Remark. The graph of $y(x) = \frac{2}{3-2x}$ has a vertical asymptote at

$$x = \frac{3}{2}.$$

Hence the function is not defined on all of $\mathbb{R}$. It is continuous on each of the intervals

$$(-\infty, \tfrac{3}{2}) \qquad \text{and} \qquad (\tfrac{3}{2}, \infty),$$

and the branch relevant to the initial condition $y(1) = 2$ is the one on the interval

$$(-\infty, \tfrac{3}{2}).$$

1.2 Separable Differential Equations and Applications

A first-order differential equation of the form

$$\frac{dy}{dx} = f(x)$$

can be solved directly by antidifferentiation, since the right-hand side depends only on the independent variable. For example, the general solution of

$$\frac{dy}{dx} = -8x$$

is

$$y(x) = \int -8x \, dx = -4x^2 + C.$$

However, if the differential equation is instead

$$\frac{dy}{dx} = -8xy,$$

then this approach no longer applies directly. In this section, we introduce a method for solving such equations called **separation of variables**. Since the right-hand side can be written as a product of a function of x and a function of y,

$$\frac{dy}{dx} = (-8x)\, y,$$

we may rewrite the equation in separated form as

$$\frac{1}{y}\,dy = -8x\,dx,$$

provided that $y \neq 0$.

Now the variables are separated, so each side can be integrated with respect to its own variable:

$$\int \frac{1}{y}\,dy = \int -8x\,dx.$$

This gives

$$\ln|y| = -4x^2 + C.$$

This relation is an *implicit solution* of the differential equation. Solving explicitly for y, we obtain

$$|y| = e^{-4x^2+C} = e^C e^{-4x^2},$$

and hence

$$y(x) = Ae^{-4x^2},$$

where A is a nonzero constant.

If we impose an initial condition, for instance $y(0) = 2$, then $A = 2$, giving the particular solution

$$y(x) = 2e^{-4x^2}.$$

Likewise, if $y(0) = -7$, then $A = -7$, so

$$y(x) = -7e^{-4x^2}.$$

It is important to note that the step of dividing by y requires $y \neq 0$. Thus the separation process produces only the nonzero solutions

$$y(x) = Ae^{-4x^2}, \qquad A \neq 0.$$

However, the constant function $y(x) \equiv 0$ also satisfies the differential equation and must be added separately. Combining both cases, we may write the complete solution set as

$$y(x) = Ae^{-4x^2}, \qquad A \in \mathbb{R}.$$

A solution such as $y(x) \equiv 0$, which may be lost during the separation process, is called a **singular solution**.

In general, a first-order differential equation is called **separable** if the function $f(x, y)$ can be written as a product of a function of x and a function of y:

$$\frac{dy}{dx} = f(x, y) = g(x)k(y) = \frac{g(x)}{h(y)},$$

where $h(y) = 1/k(y)$, assuming $k(y) \neq 0$. In this case we may rewrite the equation as

$$h(y)\,dy = g(x)\,dx,$$

or equivalently,

$$h(y)\frac{dy}{dx} = g(x).$$

If the relevant antiderivatives exist, then integrating both sides with respect to x yields

$$\int h(y(x))\frac{dy}{dx}\,dx = \int g(x)\,dx + C.$$

Define

$$H(y) = \int h(y)\,dy \qquad \text{and} \qquad G(x) = \int g(x)\,dx.$$

By the chain rule, we obtain

$$H(y) = G(x) + C.$$

Thus, under the stated assumptions, the relation

$$H(y) = G(x) + C$$

gives the **general implicit solution** of the differential equation.

Implicit, general, and singular solutions. An equation of the form $K(x, y) = 0$ is called an **implicit solution** of a differential equation if it is satisfied by some solution $y = y(x)$ of that differential equation. However, not every function determined by an implicit relation necessarily satisfies a given initial condition.

For example, consider the relation

$$x^2 + y^2 = 4.$$

Differentiating implicitly, we obtain

$$2x + 2y\frac{dy}{dx} = 0,$$

or equivalently,

$$x + y\frac{dy}{dx} = 0.$$

Thus $x^2 + y^2 = 4$ is an implicit solution of the differential equation

$$x + yy' = 0.$$

Solving explicitly for y, we find the two branches

$$y(x) = \sqrt{4 - x^2} \qquad \text{and} \qquad y(x) = -\sqrt{4 - x^2}.$$

Only the first branch satisfies the initial condition $y(0) = 2$.

In general, one should not assume that every function arising from an implicit equation satisfies the same differential equation. For instance, if we multiply the equation $x^2 + y^2 - 4 = 0$ by the factor $y - x$, we obtain

$$(y - x)(x^2 + y^2 - 4) = 0.$$

This equation yields not only the two previous solutions

$$y = \sqrt{4 - x^2} \qquad \text{and} \qquad y = -\sqrt{4 - x^2},$$

but also the additional function $y = x$, which does *not* satisfy the differential equation $x + yy' = 0$. Thus multiplying by an algebraic factor may introduce extraneous solutions.

Likewise, dividing by an algebraic factor may cause valid solutions to be lost. Consider the equation

$$(y-x)y\frac{dy}{dx} = -x(y-x).$$

The function $y(x) = x$ is clearly a solution. However, if we divide both sides by $y - x$, assuming $y \neq x$, we obtain

$$y\frac{dy}{dx} = -x,$$

or

$$x + y\frac{dy}{dx} = 0,$$

and now $y = x$ is no longer a solution of the reduced equation. Thus a valid solution has been lost by division.

A **general solution** of a differential equation is a family of solutions containing an arbitrary constant C. Each specific value of C determines a **particular solution**. In many nonlinear first-order differential equations, the complete solution set may consist of a general solution together with one or more singular solutions that cannot be obtained from any choice of the constant C.

Problem 1.2.1. *Find all solutions to the differential equation*

$$\frac{dy}{dx} = 6x(y-1)^{2/3}.$$

Solution. Assume first that $y \neq 1$. Then separation of variables gives

$$\frac{dy}{(y-1)^{2/3}} = 6x\,dx,$$

$$\int (y-1)^{-2/3}\,dy = \int 6x\,dx,$$

$$3(y-1)^{1/3} = 3x^2 + C_1.$$

Renaming the constant, we obtain

$$(y-1)^{1/3} = x^2 + C,$$

and therefore

$$y(x) = 1 + (x^2 + C)^3.$$

However, the constant function

$$y(x) \equiv 1$$

also satisfies the differential equation, and it is not obtained from the family above for any value of C. Therefore, the complete solution set consists of the one-parameter family

$$y(x) = 1 + (x^2 + C)^3, \qquad C \in \mathbb{R},$$

together with the singular solution

$$y(x) \equiv 1.$$

Natural growth and decay. The differential equation

$$\frac{dx}{dt} = kx, \qquad k \in \mathbb{R}$$

serves as a model for many naturally occurring phenomena. Equations of this type describe quantities whose time rate of change is proportional to their current size.

Population growth. Suppose $P(t)$ denotes the number of individuals in a population with constant birth rate β and constant death rate δ. Over a short time interval Δt, the number of births is approximately $\beta P(t)\Delta t$, while the number of deaths is approximately $\delta P(t)\Delta t$. Hence the net change in population is

$$\Delta P \approx (\beta - \delta)P(t)\Delta t.$$

Dividing by Δt and passing to the limit as $\Delta t \to 0$, we obtain

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = kP,$$

where

$$k = \beta - \delta.$$

The differential equation

$$\frac{dx}{dt} = kx$$

with $x(t) > 0$ is separable:

$$\int \frac{1}{x}\,dx = \int k\,dt.$$

Thus

$$\ln x = kt + C.$$

Exponentiating both sides yields

$$x(t) = e^{kt+C} = Ae^{kt},$$

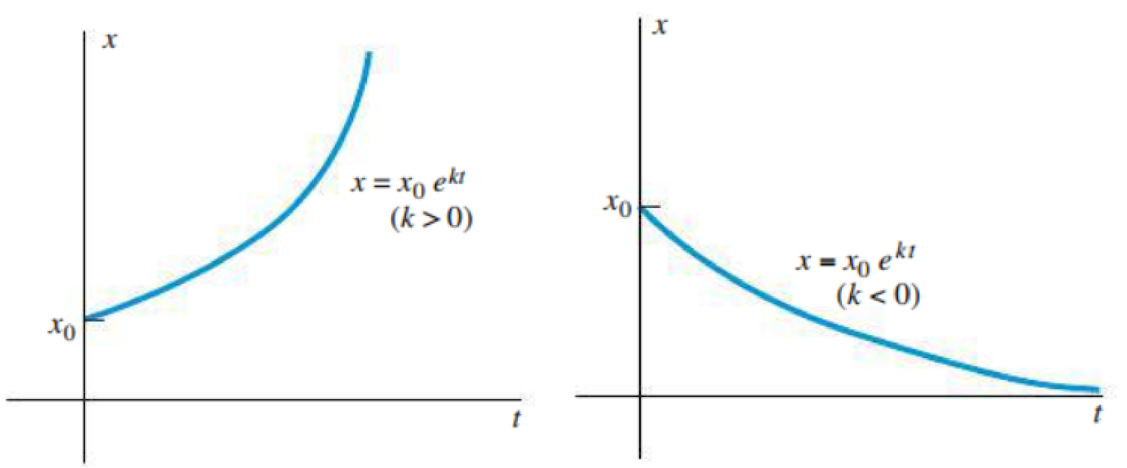
where $A = e^C$ is a positive constant. If $x(0) = x_0$, then $A = x_0$, so the solution of the initial value problem is

$$x(t) = x_0 e^{kt}.$$

Because the solution involves an exponential function, the differential equation

$$\frac{dx}{dt} = kx$$

is often called the **exponential growth equation**. If $k > 0$, the quantity grows exponentially; if $k < 0$, it decays exponentially.



Half-life of isotopes. Another important application of separable differential equations occurs in nuclear physics. If $N(t)$ denotes the amount of a radioactive substance present at time t, then it is often modeled by

$$N(t) = N_0 e^{-kt},$$

where N_0 is the initial amount and $k > 0$ is the decay constant.

The **half-life** of the isotope, denoted by τ, is the time required for half of the original sample to decay. Thus

$$N(\tau) = \frac{1}{2} N_0.$$

Substituting into the model gives

$$\frac{1}{2} N_0 = N_0 e^{-k\tau}.$$

Assuming $N_0 \neq 0$, we obtain

$$\frac{1}{2} = e^{-k\tau},$$

so

$$\tau = \frac{\ln 2}{k}.$$

For example, the half-life of the carbon isotope ^{14}C is approximately

$$\tau = \frac{\ln 2}{0.0001216} \approx 5700 \text{ years}.$$

Problem 1.2.2. *A specimen of charcoal found at Stonehenge contains* 63% *as much* ^{14}C *as a present-day sample of charcoal of equal mass. Estimate the age of the specimen.*

Solution. Let $t = 0$ denote the time at which the tree died, and let N_0 be the initial number of ^{14}C atoms in the sample. We are given that

$$N = 0.63 N_0.$$

Using the decay model

$$N = N_0 e^{-kt},$$

with $k = 0.0001216$, we obtain

$$0.63 N_0 = N_0 e^{-kt}.$$

Assuming $N_0 \neq 0$, this simplifies to

$$0.63 = e^{-kt}.$$

Taking logarithms gives

$$\ln(0.63) = -kt,$$

and therefore

$$t = -\frac{\ln(0.63)}{0.0001216} \approx \boxed{3800 \text{ years}}.$$

Cooling and heating. Recall from Section 1.1 that Newton's law of cooling states that the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant ambient temperature A is proportional to the difference $A - T$. Thus

$$\frac{dT}{dt} = k(A - T),$$

where $k > 0$.

This equation is also an example of a first-order linear differential equation. We will study linear equations systematically in the next section.

Problem 1.2.3. *A roast with initial temperature* 50°F *is placed in an oven maintained at* 375°F *at* 5:00 *P.M. After* 75 *minutes, the temperature of the roast is* 125°F*. At what time will the roast reach an internal temperature of* 150°F*?*

Solution. Let t be measured in minutes, with $t = 0$ corresponding to 5:00 P.M. We assume that the oven temperature remains constant at 375°F. Then

$$\frac{dT}{dt} = k(375 - T), \qquad T(0) = 50, \qquad T(75) = 125.$$

Separating variables, we obtain

$$\int \frac{1}{375 - T}\, dT = \int k\, dt.$$

Hence

$$-\ln|375 - T| = kt + C.$$

Since the roast temperature remains below 375°F, we have $375 - T > 0$, so the absolute value may be removed:

$$-\ln(375 - T) = kt + C.$$

Exponentiating, we get

$$375 - T = Be^{-kt}.$$

Using the initial condition $T(0) = 50$, we find

$$375 - 50 = B,$$

so $B = 325$. Therefore,

$$T(t) = 375 - 325e^{-kt}.$$

Now use the condition $T(75) = 125$:

$$125 = 375 - 325e^{-75k}.$$

Thus

$$250 = 325e^{-75k},$$

so

$$e^{-75k} = \frac{250}{325}.$$

Taking logarithms yields

$$-75k = \ln\left(\frac{250}{325}\right),$$

and hence

$$k = -\frac{1}{75}\ln\left(\frac{250}{325}\right) \approx 0.0035.$$

To determine when the roast reaches 150°F, solve

$$150 = 375 - 325e^{-0.0035t}.$$

This gives

$$225 = 325e^{-0.0035t},$$

so

$$e^{-0.0035t} = \frac{225}{325}.$$

Taking logarithms, we obtain

$$-0.0035t = \ln\left(\frac{225}{325}\right),$$

and therefore

$$t = -\frac{\ln(225/325)}{0.0035} \approx 105.$$

Thus the roast reaches 150°F about 105 minutes after 5:00 P.M., so it should be removed from the oven at approximately

$$\boxed{\text{6:45 P.M.}}$$

1.3 First-Order Linear Differential Equations

In the previous section, we studied separable differential equations and solved them by separating variables and then integrating. We now introduce another important method for solving first-order equations: the method of integrating factors.

To motivate the idea, consider the differential equation

$$\frac{dy}{dx} = 2xy, \qquad y > 0.$$

Since $y > 0$, we may divide both sides by y and obtain

$$\frac{1}{y}\frac{dy}{dx} = 2x.$$

By the chain rule,

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln y),$$

and of course

$$2x = \frac{d}{dx}(x^2).$$

Hence the differential equation may be rewritten as

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x^2).$$

Therefore the two expressions differ by a constant, and we obtain

$$\ln y = x^2 + C.$$

This example illustrates an important idea. By multiplying a differential equation by a suitably chosen nonzero function, one may transform it into an equivalent equation whose left-hand side is the derivative of a useful expression. Such a function is called an *integrating factor*.

In general, an integrating factor may depend on both the independent and dependent variables. In the example above, the factor was $1/y$, which depends only on y. In this section, however, our main goal is to study integrating factors for an especially important class of equations: first-order linear differential equations.

Definition 1.3.1. *A* first-order linear differential equation *is a differential equation of the form*

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where P and Q are given functions of x.

Throughout this section, we assume that P and Q are continuous on some interval I. Under this assumption, the equation can be solved systematically by means of an integrating factor.

Theorem 1.3.1. *Consider the linear differential equation*

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where P and Q are continuous on an interval I. Define

$$\rho(x) = e^{\int P(x)\,dx}.$$

Then $\rho(x)$ is an integrating factor for the equation, and multiplication by $\rho(x)$ transforms the equation into

$$\frac{d}{dx}\big(\rho(x)y(x)\big) = \rho(x)Q(x).$$

Consequently, the general solution is

$$y(x) = e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)\,dx}\,dx + C\right).$$

Proof. Multiply both sides of

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by

$$\rho(x) = e^{\int P(x)\,dx}.$$

This gives

$$\rho(x)\frac{dy}{dx} + P(x)\rho(x)y = \rho(x)Q(x).$$

Now

$$\rho'(x) = \frac{d}{dx}\left(e^{\int P(x)\,dx}\right) = e^{\int P(x)\,dx} \cdot \frac{d}{dx}\left(\int P(x)\,dx\right) = \rho(x)P(x),$$

so by the product rule,

$$\frac{d}{dx}\big(\rho(x)y(x)\big) = \rho(x)\frac{dy}{dx} + \rho'(x)y = \rho(x)\frac{dy}{dx} + P(x)\rho(x)y.$$

Hence

$$\frac{d}{dx}\big(\rho(x)y(x)\big) = \rho(x)Q(x).$$

Integrating both sides with respect to x, we obtain

$$\rho(x)y(x) = \int \rho(x)Q(x)\,dx + C.$$

Finally, dividing by $\rho(x) \neq 0$, we find

$$y(x) = \frac{1}{\rho(x)}\left(\int \rho(x)Q(x)\,dx + C\right) = e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)\,dx}\,dx + C\right).$$

□

Remark. When finding the integrating factor $\rho(x) = e^{\int P(x)\,dx}$, we do not include an additional constant of integration in $\int P(x)\,dx$. Indeed, any multiplicative constant in ρ is absorbed into the final arbitrary constant C appearing in the general solution.

Remark. It is better to understand the *method* than to memorize the final formula. In practice, one should identify $P(x)$ and $Q(x)$, compute the integrating factor, rewrite the left-hand side as a product derivative, and then integrate.

Problem 1.3.1. *Solve the initial value problem*

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \qquad y(0) = -1.$$

Solution. We first write the equation in the standard linear form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

from which we identify

$$P(x) = -1, \qquad Q(x) = \frac{11}{8}e^{-x/3}.$$

Therefore the integrating factor is

$$\rho(x) = e^{\int(-1)\,dx} = e^{-x}.$$

Multiplying both sides of the differential equation by e^{-x}, we obtain

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \frac{11}{8}e^{-4x/3}.$$

Since

$$\frac{d}{dx}(e^{-x}y) = e^{-x}\frac{dy}{dx} - e^{-x}y,$$

it follows that

$$\frac{d}{dx}(e^{-x}y) = \frac{11}{8}e^{-4x/3}.$$

Integrating both sides gives

$$e^{-x}y = \int \frac{11}{8}e^{-4x/3}\,dx = -\frac{33}{32}e^{-4x/3} + C.$$

Thus

$$y(x) = e^{x}\left(-\frac{33}{32}e^{-4x/3} + C\right) = Ce^{x} - \frac{33}{32}e^{-x/3}.$$

Now impose the initial condition $y(0) = -1$:

$$-1 = C - \frac{33}{32},$$

so

$$C = \frac{1}{32}.$$

Hence the solution to the initial value problem is

$$\boxed{y(x) = \frac{1}{32}\left(e^{x} - 33e^{-x/3}\right)}.$$

Problem 1.3.2. *Find the general solution of the differential equation*

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x.$$

Solution. Divide both sides by x^2+1 to obtain

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}.$$

Thus

$$P(x) = \frac{3x}{x^2+1}, \qquad Q(x) = \frac{6x}{x^2+1}.$$

The integrating factor is

$$\rho(x) = e^{\int \frac{3x}{x^2+1}\,dx}.$$

Using the substitution $u = x^2+1$, $du = 2x\,dx$, we compute

$$\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln(x^2+1).$$

Therefore

$$\rho(x) = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}.$$

Multiplying the differential equation by $(x^2+1)^{3/2}$, we obtain

$$(x^2+1)^{3/2}\frac{dy}{dx} + 3x(x^2+1)^{1/2}y = 6x(x^2+1)^{1/2}.$$

The left-hand side is the derivative of $(x^2+1)^{3/2}y$, so

$$\frac{d}{dx}\left((x^2+1)^{3/2}y\right) = 6x(x^2+1)^{1/2}.$$

Integrating both sides, we get

$$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx.$$

Again let $u = x^2+1$, $du = 2x\,dx$. Then

$$\int 6x(x^2+1)^{1/2}\,dx = 3\int u^{1/2}\,du = 2u^{3/2}+C = 2(x^2+1)^{3/2}+C.$$

Hence

$$(x^2+1)^{3/2}y = 2(x^2+1)^{3/2}+C,$$

and therefore

$$\boxed{y(x) = 2 + C(x^2+1)^{-3/2}}.$$

Application: Mixture Problems We now consider an important application of first-order linear differential equations.

Suppose a tank contains a solution consisting of a solute dissolved in a solvent. Let $x(t)$ denote the amount of solute in the tank at time t, and let $V(t)$ denote the volume of solution in the tank at time t. Assume that:

- solution flows into the tank at rate r_i liters per unit time,
- the incoming concentration is c_i units of solute per liter,
- solution flows out at rate r_o liters per unit time,
- the mixture in the tank is perfectly stirred.

During a short time interval Δt, the amount of solute entering the tank is approximately

$$r_i c_i\,\Delta t,$$

while the amount of solute leaving the tank is approximately

$$r_o c_o(t)\,\Delta t,$$

where $c_o(t)$ is the concentration in the tank at time t. Since the tank is thoroughly mixed,

$$c_o(t) = \frac{x(t)}{V(t)}.$$

Therefore,

$$\Delta x \approx r_i c_i\,\Delta t - r_o c_o(t)\,\Delta t.$$

Dividing by Δt, we get

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o(t).$$

Passing formally to the limit as $\Delta t \to 0$, we obtain the differential equation

$$\frac{dx}{dt} = r_i c_i - r_o c_o(t) = r_i c_i - \frac{r_o}{V(t)}x.$$

Hence the amount of solute satisfies

$$\boxed{\frac{dx}{dt} + \frac{r_o}{V(t)}x = r_i c_i.}$$

If $V(0) = V_0$, then

$$V(t) = V_0 + (r_i - r_o)t.$$

Thus, whenever the tank is neither empty nor overflowing, the amount $x(t)$ satisfies a first-order linear differential equation.

Problem 1.3.3. *A* 120*-gallon tank initially contains* 90 *lb of salt dissolved in* 90 *gal of water. Brine containing* 2 *lb/gal of salt flows into the tank at the rate of* 4 *gal/min, and the thoroughly mixed solution flows out at the rate of* 3 *gal/min. How much salt is in the tank when it becomes full?*

Solution. Let $x(t)$ be the amount of salt, in pounds, in the tank at time t minutes. Since liquid enters at 4 gal/min and leaves at 3 gal/min, the volume increases at the rate of 1 gal/min. Because the initial volume is 90 gal, we have

$$V(t) = 90 + t.$$

The inflow of salt is

$$(4 \text{ gal/min})(2 \text{ lb/gal}) = 8 \text{ lb/min}.$$

The outflow concentration at time t is

$$\frac{x(t)}{90+t} \text{ lb/gal},$$

so the outflow of salt is

$$3 \cdot \frac{x(t)}{90+t} \text{ lb/min}.$$

Therefore,

$$\frac{dx}{dt} = 8 - \frac{3}{90+t}x,$$

or equivalently,

$$\frac{dx}{dt} + \frac{3}{90+t}x = 8.$$

The integrating factor is

$$\rho(t) = e^{\int \frac{3}{90+t}\,dt} = e^{3\ln(90+t)} = (90+t)^3.$$

Multiplying the differential equation by $(90+t)^3$, we get

$$(90+t)^3\frac{dx}{dt} + 3(90+t)^2 x = 8(90+t)^3.$$

Thus

$$\frac{d}{dt}\left((90+t)^3 x\right) = 8(90+t)^3.$$

Integrating both sides,

$$(90+t)^3 x = \int 8(90+t)^3\,dt = 2(90+t)^4 + C.$$

Hence

$$x(t) = 2(90+t) + \frac{C}{(90+t)^3}.$$

Now use the initial condition $x(0) = 90$:

$$90 = 2(90) + \frac{C}{90^3},$$

so

$$C = -90^4.$$

Therefore,

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

The tank becomes full when its volume reaches 120 gal. Since $V(t) = 90 + t$, this occurs at

$$90 + t = 120, \qquad \text{so} \qquad t = 30.$$

Thus the amount of salt in the tank when it is full is

$$x(30) = 2(120) - \frac{90^4}{120^3} \approx 202.03125.$$

Hence the required amount is

$$\boxed{202.03 \text{ lb (approximately)}}.$$

Problem 1.3.4. *Lake Erie has volume* $480\,km^3$. *Assume that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both* $350\,km^3/yr$. *Suppose that at time* $t = 0$, *the pollutant concentration in Lake Erie is five times that in Lake Huron. Assume that all further industrial pollution ceases and that the outflow is perfectly mixed lake water. How long will it take for the pollutant concentration in Lake Erie to decrease to twice that of Lake Huron?*

Solution. Let c denote the pollutant concentration in Lake Huron, which we regard as constant. Let $x(t)$ denote the amount of pollutant in Lake Erie at time t. Since the volume of Lake Erie remains constant,

$$V = 480\,\text{km}^3.$$

Also,

$$r_i = r_o = r = 350\,\text{km}^3/\text{yr}.$$

Because the initial concentration in Lake Erie is $5c$, the initial amount of pollutant is

$$x(0) = 5cV.$$

The inflow of pollutant is rc, while the outflow of pollutant is

$$r \cdot \frac{x(t)}{V}.$$

Hence

$$\frac{dx}{dt} = rc - \frac{r}{V}x,$$

or

$$\frac{dx}{dt} + \frac{r}{V}x = rc.$$

This is a first-order linear differential equation with integrating factor

$$\rho(t) = e^{\int (r/V)\,dt} = e^{(r/V)t}.$$

Multiplying through by $\rho(t)$, we obtain

$$e^{(r/V)t}\frac{dx}{dt} + \frac{r}{V}e^{(r/V)t}x = rc\,e^{(r/V)t},$$

so that

$$\frac{d}{dt}\left(e^{(r/V)t}x\right) = rc\,e^{(r/V)t}.$$

Integrating both sides gives

$$e^{(r/V)t}x = \int rc\,e^{(r/V)t}\,dt = rc\cdot\frac{V}{r}e^{(r/V)t} + C = cVe^{(r/V)t} + C.$$

Therefore,

$$x(t) = cV + Ce^{-(r/V)t}.$$

Using $x(0) = 5cV$, we find

$$5cV = cV + C,$$

hence

$$C = 4cV.$$

So

$$x(t) = cV + 4cVe^{-rt/V}.$$

We seek the time when the concentration in Lake Erie is $2c$. Since concentration equals amount divided by volume, this means

$$x(t) = 2cV.$$

Substituting into the solution gives

$$2cV = cV + 4cVe^{-rt/V}.$$

Thus

$$1 = 4e^{-rt/V},$$

so

$$e^{-rt/V} = \frac{1}{4}.$$

Taking natural logarithms,

$$-\frac{r}{V}t = \ln\left(\frac{1}{4}\right) = -\ln 4,$$

and therefore

$$t = \frac{V}{r}\ln 4.$$

Substituting $V = 480$ and $r = 350$, we obtain

$$t = \frac{480}{350}\ln 4 \approx 1.901.$$

Hence it will take approximately

$$\boxed{1.901 \text{ years}}$$

for the pollutant concentration in Lake Erie to decrease to twice that of Lake Huron.

1.4 Substitution Methods and Exact Differential Equations

So far, we have studied first-order differential equations that are either separable or linear. However, many first-order equations do not initially appear in either of these forms. In such cases, a well-chosen substitution can often transform the equation into one that is easier to solve.

Suppose we are given a first-order differential equation

$$\frac{dy}{dx} = f(x, y).$$

Sometimes the expression on the right-hand side suggests a new variable

$$v = \alpha(x, y),$$

which allows us to rewrite the equation in terms of x and v. If the substitution can be solved locally for y, say

$$y = \beta(x, v),$$

then by the Chain Rule,

$$\frac{dy}{dx} = \frac{\partial \beta}{\partial x} + \frac{\partial \beta}{\partial v}\frac{dv}{dx}.$$

Substituting this expression into the original differential equation may produce a new equation involving v and x. If the transformed equation is separable or linear, then the methods from the previous sections apply.

The main difficulty is choosing a useful substitution. In many examples, the appropriate substitution is suggested by the structure of the equation itself.

Problem 1.4.1. *Solve the differential equation*

$$\frac{dy}{dx} = (x + y + 3)^2.$$

Solution. The quantity $x + y + 3$ appears as a single unit, so we introduce the substitution

$$v = x + y + 3.$$

Then

$$y = v - x - 3,$$

and differentiating with respect to x gives

$$\frac{dy}{dx} = \frac{dv}{dx} - 1.$$

Substituting into the differential equation, we obtain

$$\frac{dv}{dx} - 1 = v^2,$$

or equivalently,

$$\frac{dv}{dx} = 1 + v^2.$$

This equation is separable:

$$\frac{dv}{1 + v^2} = dx.$$

Integrating both sides yields

$$\arctan v = x + C.$$

Hence

$$v = \tan(x + C).$$

Since $v = x + y + 3$, we conclude that

$$x + y + 3 = \tan(x + C),$$

and therefore

$$\boxed{y(x) = \tan(x + C) - x - 3.}$$

More generally, any differential equation of the form

$$\frac{dy}{dx} = F(ax + by + c)$$

can be reduced to a separable equation by the substitution

$$v = ax + by + c.$$

Homogeneous Equations

Definition 1.4.1. *A first-order differential equation is called* homogeneous *if it can be written in the form*

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right),$$

where $x \neq 0$.

For a homogeneous equation, the substitution

$$v = \frac{y}{x}$$

is natural. Since $y = vx$, the Product Rule gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting into the differential equation

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

yields

$$v + x\frac{dv}{dx} = F(v),$$

so that

$$x\frac{dv}{dx} = F(v) - v.$$

This is a separable equation. Thus every homogeneous first-order differential equation can be reduced to a separable equation by the substitution $v = y/x$.

Remark. There is another common characterization of homogeneous equations. Suppose $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of the same degree. Then the differential equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$$

is homogeneous, since the quotient depends only on y/x. Indeed, dividing numerator and denominator by the appropriate power of x shows that the right-hand side can be expressed entirely as a function of y/x.

Problem 1.4.2. *Solve the differential equation*

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2.$$

Solution. First divide both sides by $2xy$ to obtain

$$\frac{dy}{dx} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}.$$

This is a homogeneous equation, since the right-hand side depends only on the ratio y/x. Let

$$v = \frac{y}{x}, \qquad \text{so that} \qquad y = vx$$

and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting gives

$$v + x\frac{dv}{dx} = \frac{2}{v} + \frac{3}{2}v.$$

Therefore,

$$x\frac{dv}{dx} = \frac{2}{v} + \frac{1}{2}v = \frac{v^2 + 4}{2v}.$$

Separating variables,

$$\frac{2v}{v^2 + 4}\, dv = \frac{dx}{x}.$$

Integrating,

$$\int \frac{2v}{v^2 + 4}\, dv = \int \frac{1}{x}\, dx,$$

so

$$\ln(v^2 + 4) = \ln|x| + C.$$

Exponentiating,

$$v^2 + 4 = C|x|.$$

Now substitute $v = y/x$:

$$\frac{y^2}{x^2} + 4 = C|x|.$$

Multiplying by x^2, we obtain

$$y^2 + 4x^2 = C|x|^3.$$

On any interval where $x > 0$, we may absorb the sign into the constant and write

$$\boxed{y^2 + 4x^2 = Cx^3.}$$

Similarly, on intervals where $x < 0$, the constant can again be adjusted appropriately.

Problem 1.4.3. *Solve the initial value problem for $x_0 > 0$:*

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \qquad y(x_0) = 0.$$

Solution. Divide both sides by x to obtain

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}.$$

This is homogeneous. Let

$$v = \frac{y}{x}, \qquad \text{so that} \qquad y = vx$$

and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting into the differential equation yields

$$v + x\frac{dv}{dx} = v + \sqrt{1 - v^2}.$$

Hence

$$x\frac{dv}{dx} = \sqrt{1 - v^2}.$$

Separating variables gives

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x}.$$

Since $x_0 > 0$, we work on an interval where $x > 0$, so

$$\int \frac{dv}{\sqrt{1 - v^2}} = \int \frac{dx}{x}.$$

Thus

$$\arcsin v = \ln x + C.$$

Using the initial condition $y(x_0) = 0$, we find

$$v(x_0) = \frac{y(x_0)}{x_0} = 0.$$

Therefore,

$$0 = \arcsin(0) = \ln x_0 + C,$$

so

$$C = -\ln x_0.$$

Hence

$$\arcsin v = \ln x - \ln x_0 = \ln\left(\frac{x}{x_0}\right),$$

and therefore

$$v = \sin\left(\ln\frac{x}{x_0}\right).$$

Since $v = y/x$, we conclude that

$$\boxed{y(x) = x\sin\left(\ln\frac{x}{x_0}\right).}$$

Bernoulli Differential Equations

Definition 1.4.2. *A* Bernoulli differential equation *is a first-order differential equation of the form*

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where $n \neq 0, 1$.

When $n = 0$ or $n = 1$, the equation is already linear. For $n \neq 0, 1$, the substitution

$$v = y^{1-n}$$

transforms the equation into a linear first-order differential equation.

Indeed, differentiating $v = y^{1-n}$ gives

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Multiplying the Bernoulli equation by $(1-n)y^{-n}$, one obtains

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which is linear in the new dependent variable v.

Problem 1.4.4. *Find the general solution of the differential equation*

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}.$$

Solution. First divide by x:

$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}.$$

This is a Bernoulli equation with

$$n = \frac{4}{3}.$$

Hence

$$1 - n = -\frac{1}{3},$$

so we use the substitution

$$v = y^{-1/3}.$$

Then

$$\frac{dv}{dx} = -\frac{1}{3}y^{-4/3}\frac{dy}{dx}.$$

Equivalently,

$$\frac{dy}{dx} = -3y^{4/3}\frac{dv}{dx}.$$

Substituting into the original equation gives

$$x\left(-3y^{4/3}\frac{dv}{dx}\right) + 6y = 3xy^{4/3}.$$

Now divide through by $y^{4/3}$:

$$-3x\frac{dv}{dx} + 6y^{-1/3} = 3x.$$

Since $v = y^{-1/3}$, this becomes

$$-3x\frac{dv}{dx} + 6v = 3x.$$

Dividing by $-3x$, we obtain the linear equation

$$\frac{dv}{dx} - \frac{2}{x}v = -1.$$

The integrating factor is

$$\mu(x) = e^{\int -2/x\,dx} = x^{-2}.$$

Multiplying through by x^{-2}, we get

$$x^{-2}\frac{dv}{dx} - 2x^{-3}v = -x^{-2},$$

so

$$\frac{d}{dx}\left(x^{-2}v\right) = -x^{-2}.$$

Integrating,

$$x^{-2}v = \int -x^{-2}\,dx = \frac{1}{x} + C.$$

Thus

$$v = x + Cx^2.$$

Finally, since $v = y^{-1/3}$,

$$y = (x + Cx^2)^{-3}.$$

Therefore,

$$\boxed{y(x) = \frac{1}{(x + Cx^2)^3}.}$$

Problem 1.4.5. *Solve the differential equation*

$$2xe^{2y}\frac{dy}{dx} = 3x^4 + e^{2y}.$$

Solution. Observe that y appears only through the expression e^{2y}, and

$$\frac{d}{dx}(e^{2y}) = 2e^{2y}\frac{dy}{dx}.$$

This suggests the substitution

$$v = e^{2y}.$$

Then

$$\frac{dv}{dx} = 2e^{2y}\frac{dy}{dx},$$

so the differential equation becomes

$$x\frac{dv}{dx} = 3x^4 + v.$$

Rewriting,

$$\frac{dv}{dx} - \frac{1}{x}v = 3x^3.$$

This is linear in v. The integrating factor is

$$\mu(x) = e^{\int -1/x\,dx} = x^{-1}.$$

Multiplying through by x^{-1} gives

$$x^{-1}\frac{dv}{dx} - x^{-2}v = 3x^2,$$

hence

$$\frac{d}{dx}\left(\frac{v}{x}\right) = 3x^2.$$

Integrating,

$$\frac{v}{x} = x^3 + C,$$

so

$$v = x^4 + Cx.$$

Since $v = e^{2y} > 0$, we must have

$$x^4 + Cx > 0$$

on the interval of definition. Therefore,

$$e^{2y} = x^4 + Cx,$$

and

$$\boxed{y(x) = \frac{1}{2}\ln(x^4 + Cx),}$$

where the expression inside the logarithm is positive.

Exact Differential Equations Many first-order differential equations admit an implicit solution of the form

$$F(x, y) = C,$$

where F is a differentiable function and C is an arbitrary constant. Differentiating both sides with respect to x, where $y = y(x)$, gives

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

If we define

$$M(x, y) = \frac{\partial F}{\partial x}, \qquad N(x, y) = \frac{\partial F}{\partial y},$$

then the differential equation takes the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Multiplying by dx, we obtain the *differential form*

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

Conversely, given an equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

we ask whether there exists a function $F(x, y)$ such that

$$F_x = M \qquad \text{and} \qquad F_y = N.$$

If such a function exists, then

$$dF = F_x\,dx + F_y\,dy = M\,dx + N\,dy,$$

and the equation is called *exact*. In that case, the implicit relation

$$F(x, y) = C$$

gives the general solution.

The condition for exactness comes from equality of mixed partial derivatives. If F_{xy} and F_{yx} are continuous, then Clairaut's Theorem implies

$$F_{xy} = F_{yx}.$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(F_x) = F_{xy} = F_{yx} = \frac{\partial}{\partial x}(F_y) = \frac{\partial N}{\partial x}.$$

Definition 1.4.3. *A differential equation of the form*

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called exact *on a region R if there exists a function $F(x, y)$ on R such that*

$$F_x = M \qquad \textit{and} \qquad F_y = N.$$

A necessary condition for exactness is

$$M_y = N_x.$$

The next theorem gives a convenient criterion for exactness.

Theorem 1.4.1. *Suppose that $M(x, y)$ and $N(x, y)$ are continuous on a rectangle*

$$R = \{(x, y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\},$$

and that their first-order partial derivatives M_y and N_x are also continuous on R. Then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact on R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

at every point of R.

Proof. We have already shown that exactness implies $M_y = N_x$. For the converse, assume that

$$M_y = N_x$$

throughout R. We will construct a function $F(x, y)$ such that $F_x = M$ and $F_y = N$.

Define

$$F(x, y) = \int M(x, y)\, dx + g(y),$$

where $g(y)$ is to be determined. By construction,

$$F_x = M.$$

Now differentiate with respect to y:

$$F_y = \frac{\partial}{\partial y}\left(\int M(x, y)\, dx\right) + g'(y).$$

To ensure that $F_y = N$, we require

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\left(\int M(x, y)\, dx\right).$$

It remains to show that the right-hand side depends only on y. Differentiate with respect to x:

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\left(\int M(x, y)\, dx\right)\right] = N_x - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)\, dx\right).$$

Since

$$\frac{\partial}{\partial x}\int M(x, y)\, dx = M(x, y),$$

this becomes

$$N_x - M_y = 0$$

by hypothesis. Hence the quantity

$$N(x, y) - \frac{\partial}{\partial y}\left(\int M(x, y)\, dx\right)$$

is independent of x, so it is indeed a function only of y. Therefore $g(y)$ exists, and the equation is exact. □

As a simple example, consider

$$y^3\, dx + 3xy^2\, dy = 0.$$

Here

$$M(x, y) = y^3, \qquad N(x, y) = 3xy^2.$$

By inspection, the function

$$F(x, y) = xy^3$$

satisfies

$$F_x = y^3 = M \qquad \text{and} \qquad F_y = 3xy^2 = N.$$

Hence the equation is exact, and its implicit solution is

$$xy^3 = C.$$

Solving for y, we obtain

$$\boxed{y(x) = kx^{-1/3},}$$

where k is an arbitrary constant.

Problem 1.4.6. *Determine whether the differential equation*

$$(2x\cos 2y + \sin 2x)\frac{dy}{dx} + \sin 2y + 2y\cos 2x = 0$$

is exact.

Solution. Write the equation in differential form:

$$(\sin 2y + 2y\cos 2x)\,dx + (2x\cos 2y + \sin 2x)\,dy = 0.$$

Thus

$$M(x,y) = \sin 2y + 2y\cos 2x, \qquad N(x,y) = 2x\cos 2y + \sin 2x.$$

Now compute the partial derivatives:

$$M_y = 2\cos 2y + 2\cos 2x,$$

and

$$N_x = 2\cos 2y + 2\cos 2x.$$

Since $M_y = N_x$, the equation is exact.

Problem 1.4.7. *Solve the differential equation*

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

after verifying that it is exact.

Solution. Let

$$M(x,y) = 6xy - y^3, \qquad N(x,y) = 4y + 3x^2 - 3xy^2.$$

First compute

$$M_y = 6x - 3y^2, \qquad N_x = 6x - 3y^2.$$

Since $M_y = N_x$, the equation is exact.

Now find a potential function $F(x,y)$ satisfying

$$F_x = M.$$

Integrating M with respect to x,

$$F(x,y) = \int (6xy - y^3)\,dx = 3x^2y - xy^3 + g(y),$$

where $g(y)$ is an unknown function of y.

Differentiate with respect to y:

$$F_y = 3x^2 - 3xy^2 + g'(y).$$

Since $F_y = N$, we must have

$$3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2.$$

Thus

$$g'(y) = 4y,$$

so

$$g(y) = 2y^2.$$

Therefore,

$$F(x, y) = 3x^2y - xy^3 + 2y^2.$$

The implicit general solution is

$$\boxed{3x^2y - xy^3 + 2y^2 = C.}$$

2 Linear Systems and Matrices

2.1 Introduction to Linear Systems

In many areas of mathematics, science, and engineering, one is led naturally to the study of collections of linear equations in several unknowns. Such collections are called *linear systems.* They arise whenever several quantities are constrained by linear relationships, and they serve as one of the most fundamental objects in algebra. The theory of linear systems is important not only because of its direct applications, but also because it provides the conceptual starting point for matrices, determinants, vector spaces, and linear transformations.

A typical linear system in the unknowns $x_1, x_2, \ldots, x_n$ consists of equations of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \qquad i = 1, 2, \ldots, m,$$

where the coefficients a_{ij} and constants b_i are scalars, usually taken to lie in $\mathbb{R}$ or $\mathbb{C}$. Here m denotes the number of equations, while n denotes the number of unknowns. The central problem is to determine all n-tuples $(x_1, x_2, \ldots, x_n)$ that satisfy every equation in the system simultaneously.

Definition 2.1.1. *A* linear equation *in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are scalars. A linear system *is a finite collection of such equations.*

The term "linear" reflects the fact that each variable appears only to the first power, and no products of variables occur. Thus equations such as

$$2x - 3y + z = 5$$

are linear, whereas equations such as

$$x^2 + y = 1, \qquad xy + z = 4$$

are not linear.

Example 2.1.1. *The system*

$$\begin{aligned} x + 2y - z &= 3, \\ 2x - y + 3z &= 7, \\ 3x + y + 2z &= 4 \end{aligned}$$

is a linear system in the unknowns x, y, z.

Definition 2.1.2. *A* solution *of a linear system is an ordered n-tuple $(s_1, s_2, \ldots, s_n)$ such that substituting $x_j = s_j$ for each j satisfies every equation in the system.*

To solve a linear system means to determine its entire solution set. In principle, a system may behave in one of several ways: it may have exactly one solution, it may have infinitely many solutions, or it may have no solution at all.

Example 2.1.2. *Consider the system*

$$\begin{aligned} x + y &= 2, \\ x - y &= 0. \end{aligned}$$

Adding the two equations yields $2x = 2$*, so* $x = 1$*. Substituting into the first equation gives* $y = 1$*. Hence the system has the unique solution* $(1, 1)$*.*

Example 2.1.3. *Consider the system*

$$\begin{aligned} x + y + z &= 2, \\ 2x + 2y + 2z &= 4. \end{aligned}$$

The second equation is simply twice the first, so the two equations impose the same condition. Consequently, there are infinitely many solutions, namely all triples (x, y, z) *satisfying*

$$x + y + z = 2.$$

Example 2.1.4. *Consider the system*

$$\begin{aligned} x + y &= 1, \\ x + y &= 3. \end{aligned}$$

No ordered pair (x, y) *can satisfy both equations simultaneously. Hence the system has no solution.*

These examples illustrate the three fundamental possibilities for a linear system. It is therefore natural to introduce the following terminology.

Definition 2.1.3. *A linear system is said to be* consistent *if it has at least one solution, and* inconsistent *if it has no solution.*

Thus a consistent system may have either one solution or infinitely many solutions, whereas an inconsistent system has none. One of the main objectives in the study of linear systems is to determine efficiently which of these cases occurs for a given system.

Equivalent Systems In solving a linear system, one often replaces it by another system that is easier to analyze but has exactly the same solutions. This motivates the notion of equivalence.

Definition 2.1.4. *Two linear systems are called* equivalent *if they have the same solution set.*

The usefulness of equivalent systems lies in the fact that one may transform a complicated system into a simpler one without changing the set of solutions. For instance, the systems

$$\begin{aligned} x + y &= 2, \\ x - y &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + y &= 2, \\ 2x &= 2 \end{aligned}$$

are equivalent, since the second system is obtained from the first by adding the two equations, and both systems have the same unique solution $(1, 1)$.

More generally, experience suggests that certain operations on the equations of a system preserve equivalence. For example, interchanging two equations clearly does not affect the set of solutions, since the order in which equations are written is irrelevant. Likewise, multiplying an equation by a nonzero scalar does not change its set of solutions, and adding a multiple of one equation to another often simplifies the system while preserving equivalence. These operations will play a central role in what follows.

Why Linear Systems Matter At first glance, linear systems may seem to be merely collections of algebraic equations. However, they encode a remarkably broad range of mathematical questions. Geometrically, a single linear equation in two variables describes a line in the plane, while a linear equation in three variables describes a plane in space. Thus solving a linear system amounts to determining the intersection of several geometric objects. In applications, linear systems arise in circuit analysis, chemical balancing, economics, statistics, computer graphics, and the numerical solution of differential equations.

From a theoretical standpoint, linear systems are the gateway to much of linear algebra. The methods developed for solving them will lead naturally to matrices, whose compact notation allows one to organize the coefficients of a system efficiently. Once matrices are introduced, one can study row reduction systematically, define matrix operations, characterize invertibility, and eventually relate these ideas to determinants and linear transformations.

The Need for Systematic Methods Although small systems can sometimes be solved by substitution or elimination by hand, such ad hoc methods quickly become impractical when the number of equations and unknowns grows. Even for moderately sized systems, one needs a systematic procedure that can be carried out efficiently and, ideally, implemented algorithmically.

The classical method that addresses this need is *Gaussian elimination*. Rather than working informally with equations, Gaussian elimination organizes the coefficients of a system into an array and performs a sequence of elementary operations to simplify the system step by step. This process not only makes computations more efficient, but also reveals the underlying structure of the solution set.

For this reason, the next section introduces *matrices* as convenient devices for recording linear systems and develops the method of Gaussian elimination in a precise and systematic way. That transition is natural: once the primary goal is to solve linear systems efficiently, matrices provide exactly the language and framework needed to do so.

2.2 Matrices and Gaussian Elimination

In the previous section, we introduced systems of linear equations and discussed the basic problem of determining whether such systems possess solutions, and if so, whether those solutions are unique. In practice, solving linear systems efficiently requires a more systematic method than repeated substitution, especially when the number of equations and unknowns becomes large. The language of *matrices* provides precisely such a framework.

The purpose of this section is twofold. First, we introduce matrices as convenient devices for encoding linear systems. Second, we develop the method of *Gaussian elimination*, which transforms a given system into a simpler equivalent one from which the solution can be read more readily. This method is one of the foundational algorithms of linear algebra and will reappear throughout the subject in many different guises.

Matrices Associated with Linear Systems Consider the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3, \\ 2x_1 - x_2 + 3x_3 &= 7, \\ -x_1 + 4x_2 + 2x_3 &= 1. \end{aligned}$$

A system such as this contains two kinds of data: the coefficients of the unknowns and the constants on the right-hand side. It is often convenient to record only these numbers in rectangular arrays.

Definition 2.2.1. *A* matrix *is a rectangular array of numbers arranged in rows and columns. If a matrix has m rows and n columns, then it is said to be an $m \times n$ matrix.*

The matrix of coefficients of the above system is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 4 & 2 \end{pmatrix},$$

while the column of constants is

$$\mathbf{b} = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}.$$

Together these form the *augmented matrix*

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 7 \\ -1 & 4 & 2 & 1 \end{array}\right].$$

Definition 2.2.2. *Given a linear system, the* augmented matrix *of the system is the matrix obtained by adjoining the column of constants to the coefficient matrix.*

The augmented matrix contains all the numerical information needed to solve the system. The variables themselves may be temporarily suppressed, since they can always be restored later.

Example 2.2.1. *Write the augmented matrix of the system*

$$\begin{aligned} 2x - y + 3z &= 5, \\ 4x + 2y - z &= 1. \end{aligned}$$

Solution. The coefficient matrix is

$$\begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix},$$

and the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 4 & 2 & -1 & 1 \end{array}\right].$$

Problem 2.2.1. *Write the augmented matrix for each system:*

1.

$$\begin{aligned} x + 2y &= 4, \\ 3x - y &= 5; \end{aligned}$$

2.

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0, \\ x_2 + 4x_3 &= 7, \\ 5x_1 - 2x_3 &= -1. \end{aligned}$$

Solution.

1. The coefficient matrix is
$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$$
and the augmented matrix is
$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -1 & 5 \end{array}\right].$$

2. Here, the coefficient matrix is
$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \\ 5 & 0 & -2 \end{pmatrix}$$
and the augmented matrix is
$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 1 & 4 & 7 \\ 5 & 0 & -2 & -1 \end{array}\right].$$

Elementary Row Operations The central idea behind elimination is that certain operations may be performed on the equations of a system without changing its solution set. Since rows of an augmented matrix correspond to equations, these same operations may be applied directly to the rows of the matrix.

Definition 2.2.3. *The following operations on the rows of a matrix are called* elementary row operations*:*

1. *Interchange two rows.*
2. *Multiply a row by a nonzero scalar.*
3. *Add a scalar multiple of one row to another row.*

Each of these operations preserves the set of solutions of the corresponding linear system. Thus, if one augmented matrix is obtained from another by a sequence of elementary row operations, then the two corresponding systems are *equivalent*.

Example 2.2.2. *Consider the system*
$$\begin{aligned} x + y &= 2, \\ 2x + 3y &= 5. \end{aligned}$$
Its augmented matrix is
$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right).$$
Replacing the second row by
$$R_2 - 2R_1$$
gives
$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right).$$

This corresponds to the equivalent system

$$\begin{aligned} x + y &= 2, \\ y &= 1. \end{aligned}$$

From this, we immediately obtain $x = 1$.

Problem 2.2.2. *For each of the following, perform the indicated row operation:*

1. $$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right), \qquad R_2 \leftarrow R_2 - 4R_1.$$

2. $$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \\ 2 & 3 & 1 & 5 \end{array}\right), \qquad R_3 \leftarrow R_3 - 2R_1.$$

3. $$\left(\begin{array}{cc|c} 2 & -1 & 7 \\ 3 & 4 & 1 \end{array}\right), \qquad R_1 \leftrightarrow R_2.$$

Solution.

1. $$\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 6 \end{pmatrix}, \qquad R_2 \leftarrow R_2 - 4R_1$$

   Compute the new second row:
   $$R_2 = (4, 5, 6) - 4(1, 2, 3) = (4 - 4,\ 5 - 8,\ 6 - 12) = (0, -3, -6).$$
   Thus the matrix becomes
   $$\begin{pmatrix} 1 & 2 & | & 3 \\ 0 & -3 & | & -6 \end{pmatrix}.$$

2. $$\begin{pmatrix} 1 & 0 & 2 & | & 3 \\ 0 & 1 & -1 & | & 4 \\ 2 & 3 & 1 & | & 5 \end{pmatrix}, \qquad R_3 \leftarrow R_3 - 2R_1$$

   Compute the new third row:
   $$R_3 = (2, 3, 1, 5) - 2(1, 0, 2, 3) = (2 - 2,\ 3 - 0,\ 1 - 4,\ 5 - 6) = (0, 3, -3, -1).$$
   Thus the matrix becomes
   $$\begin{pmatrix} 1 & 0 & 2 & | & 3 \\ 0 & 1 & -1 & | & 4 \\ 0 & 3 & -3 & | & -1 \end{pmatrix}.$$

3. $$\begin{pmatrix} 2 & -1 & | & 7 \\ 3 & 4 & | & 1 \end{pmatrix}, \qquad R_1 \leftrightarrow R_2$$

   Interchanging the two rows gives
   $$\begin{pmatrix} 3 & 4 & | & 1 \\ 2 & -1 & | & 7 \end{pmatrix}.$$

The Method of Gaussian Elimination The goal of Gaussian elimination is to transform the augmented matrix of a system into a simpler form, usually one in which many entries below leading coefficients are zero. This triangular structure allows the system to be solved by back-substitution.

Definition 2.2.4. *A nonzero entry in a row is called a* leading entry *if it is the first nonzero entry from the left in that row.*

In Gaussian elimination, one attempts to create zeros below each leading entry. This produces what is often called *row-echelon form*, which we will study more formally in the next section.

Example 2.2.3. *Solve the system*

$$\begin{aligned} x + 2y - z &= 3, \\ 2x - y + 3z &= 7, \\ -x + 4y + 2z &= 1. \end{aligned}$$

Solution. We begin with the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 7 \\ -1 & 4 & 2 & 1 \end{array}\right).$$

Use the first row to eliminate the entries below the leading 1 in column one:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 + R_1.$$

This yields

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & 1 \\ 0 & 6 & 1 & 4 \end{array}\right).$$

Next eliminate the entry below the leading entry in the second row. One convenient choice is

$$R_3 \leftarrow 5R_3 + 6R_2.$$

Then the matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & 1 \\ 0 & 0 & 35 & 26 \end{array}\right).$$

Inspecting the third row, we have

$$35z = 26 \quad \Longrightarrow \quad z = \frac{26}{35}.$$

From the second row,

$$-5y + 5z = 1,$$

so

$$-5y + 5\left(\frac{26}{35}\right) = 1.$$

Hence

$$-5y + \frac{26}{7} = 1 \quad \Longrightarrow \quad -5y = -\frac{19}{7} \quad \Longrightarrow \quad y = \frac{19}{35}.$$

Finally, the first row gives

$$x + 2y - z = 3,$$

so substituting from the previous results yields

$$x + 2\left(\frac{19}{35}\right) - \frac{26}{35} = 3.$$

Thus

$$x + \frac{12}{35} = 3 \quad \Longrightarrow \quad x = \frac{93}{35}.$$

Therefore the unique solution is

$$(x, y, z) = \left(\frac{93}{35}, \frac{19}{35}, \frac{26}{35}\right).$$

Remark. In actual computation, many different sequences of row operations are possible. Some are more efficient than others, but any valid sequence of elementary row operations leading to a solved form will produce the same solution set.

Problem 2.2.3. *Use Gaussian elimination to solve the following systems:*

1.

$$\begin{aligned} x + y &= 3, \\ 2x + 3y &= 8. \end{aligned}$$

2.

$$\begin{aligned} x + 2y - z &= 1, \\ 2x + 5y + z &= 4, \\ -x + y + 2z &= 0. \end{aligned}$$

Solution.

1. We are given

$$\begin{cases} x + y = 3, \\ 2x + 3y = 8. \end{cases}$$

First we write the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 3 & 8 \end{array}\right).$$

Then perform the row operation $R_2 \leftarrow R_2 - 2R_1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right).$$

From the second row we get $y = 2$. Substitute this into the first equation:

$$x + 2 = 3 \quad \Rightarrow \quad x = 1.$$

Therefore, the solution to the system is

$$\boxed{(x, y) = (1, 2).}$$

2. We are given

$$\begin{cases} x + 2y - z = 1, \\ 2x + 5y + z = 4, \\ -x + y + 2z = 0. \end{cases}$$

Write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 1 & 4 \\ -1 & 1 & 2 & 0 \end{array}\right).$$

Perform $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 + R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 1 & 1 \end{array}\right).$$

Next perform $R_3 \leftarrow R_3 - 3R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -8 & -5 \end{array}\right).$$

From the third row:

$$-8z = -5 \quad \Rightarrow \quad z = \frac{5}{8}.$$

Substitute into the second row:

$$y + 3\left(\frac{5}{8}\right) = 2 \quad \Rightarrow \quad y = \frac{1}{8}.$$

Substitute y and z into the first row:

$$x + 2\left(\frac{1}{8}\right) - \frac{5}{8} = 1 \quad \Rightarrow \quad x = \frac{11}{8}.$$

Therefore, the solution is

$$\boxed{(x, y, z) = \left(\frac{11}{8}, \frac{1}{8}, \frac{5}{8}\right).}$$

Back-Substitution Once a system has been reduced to triangular form, its solution is obtained by solving first for the last variable, then substituting upward step by step. This procedure is called *back-substitution.*

Example 2.2.4. *Consider the system*

$$\begin{aligned} x + 2y - z &= 4, \\ y + 3z &= 5, \\ 2z &= 6. \end{aligned}$$

The third equation gives $z = 3$. Substituting into the second equation yields

$$y + 3(3) = 5 \quad \Longrightarrow \quad y = -4.$$

Substituting both values into the first equation gives

$$x + 2(-4) - 3 = 4,$$

so $x = 15$. *Therefore the solution is*

$$(x, y, z) = (15, -4, 3).$$

Problem 2.2.4. *Solve by back-substitution:*

$$\begin{aligned} x - y + 2z &= 1, \\ + 3y - z &= 7, \\ + 4z &= 8. \end{aligned}$$

Solution. We solve the system by back-substitution:

$$\begin{aligned} x - y + 2z &= 1, \\ 3y - z &= 7, \\ 4z &= 8. \end{aligned}$$

Start with the last equation:

$$4z = 8 \quad \Rightarrow \quad z = 2.$$

Substitute $z = 2$ into the second equation:

$$3y - z = 7 \Rightarrow 3y - 2 = 7 \Rightarrow 3y = 9 \Rightarrow y = 3.$$

Now substitute $y = 3$ and $z = 2$ into the first equation:

$$x - y + 2z = 1 \Rightarrow x - 3 + 4 = 1 \Rightarrow x + 1 = 1 \Rightarrow x = 0.$$

Thus the solution to the system is

$$\boxed{(x, y, z) = (0, 3, 2).}$$

Systems with No Solution or Infinitely Many Solutions Gaussian elimination does more than merely compute solutions. It also reveals whether a system is inconsistent or whether it has free variables.

Example 2.2.5 (An inconsistent system)**.** *Consider*

$$\begin{aligned} x + y &= 2, \\ 2x + 2y &= 5. \end{aligned}$$

Its augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right).$$

Applying

$$R_2 \leftarrow R_2 - 2R_1$$

gives

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right).$$

The second row corresponds to the equation

$$0 = 1,$$

which is impossible. Hence the system has no solution.

Example 2.2.6 (A dependent system)**.** *Now consider*

$$\begin{aligned} x + y &= 2, \\ 2x + 2y &= 4. \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right).$$

Again applying

$$R_2 \leftarrow R_2 - 2R_1$$

gives

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

This corresponds to the single equation

$$x + y = 2.$$

Since one variable may be chosen freely, there are infinitely many solutions. If we let $y = t$, then

$$x = 2 - t.$$

Thus the solution set is

$$(x, y) = (2 - t, t), \qquad t \in \mathbb{R}.$$

Problem 2.2.5. *Determine whether each system has no solution, a unique solution, or infinitely many solutions:*

1. $$\begin{aligned} x + y &= 1, \\ 2x + 2y &= 2; \end{aligned}$$

2. $$\begin{aligned} x + y &= 1, \\ 2x + 2y &= 3; \end{aligned}$$

3. $$\begin{aligned} x + y + z &= 2, \\ 2x + 2y + 2z &= 4, \\ x - y + z &= 0. \end{aligned}$$

Solution.

1. The system is
$$x + y = 1, \qquad 2x + 2y = 2.$$
The second equation is twice the first equation, so both equations represent the same line. Hence the system is dependent and has **infinitely many solutions.**
Let $x = t$. Then from $x + y = 1$,
$$y = 1 - t.$$
Thus the solution set is
$$(x, y) = (t, 1 - t), \qquad t \in \mathbb{R}.$$
Therefore, the system has infinitely many solutions.

2. The system is
$$x + y = 1, \qquad 2x + 2y = 3.$$
Multiply the first equation by 2:
$$2x + 2y = 2.$$
But the second equation states
$$2x + 2y = 3.$$
This gives the contradiction
$$2 = 3.$$
Hence the system is inconsistent and has **no solution.**

3. The corresponding linear system is
$$x + y + z = 2,$$
$$2x + 2y + 2z = 4,$$
$$x - y + z = 0.$$
The second equation is twice the first, so it does not provide any new useful information. We solve utilizing the first and third equations.
From the third equation,
$$x - y + z = 0 \quad \Rightarrow \quad y = x + z.$$
Substitute into the first equation:
$$x + (x + z) + z = 2.$$
Simplifying gives
$$2x + 2z = 2 \quad \Rightarrow \quad x + z = 1.$$
Let $z = t$. Then
$$x = 1 - t.$$

Using $y = x + z$,

$$y = (1 - t) + t = 1.$$

Thus the solution set is

$$(x, y, z) = (1 - t,\ 1,\ t), \qquad t \in \mathbb{R}.$$

Therefore, the system has **infinitely many solutions.**

A General Procedure The method of Gaussian elimination may be summarized as follows.

1. Write the augmented matrix of the system.
2. Use elementary row operations to create zeros below leading entries.
3. Continue until the matrix is in echelon-like form.
4. Interpret the resulting system:
   a) If a contradiction appears, the system is inconsistent.
   b) If every variable is determined uniquely, the system has a unique solution.
   c) If one or more variables are free, the system has infinitely many solutions.
5. When appropriate, use back-substitution to solve for the variables.

This algorithm is effective not only by hand for small systems, but also as the basis for machine computation in more complicated problems.

Why the Method Works Although the computational steps of elimination are straightforward, it is worth emphasizing the conceptual reason for their validity: each elementary row operation produces a system equivalent to the original one. Thus, throughout the elimination process, we are not altering the actual solution set; we are merely rewriting the system in progressively simpler forms.

This viewpoint is fundamental in linear algebra. Much of the subject involves replacing a complicated object by a simpler equivalent one without losing the essential information. Gaussian elimination is the first major example of this principle.

Exercises

1. Solve the system

$$\begin{aligned} x + 2y &= 5, \\ 3x - y &= 4. \end{aligned}$$

2. Solve the system

$$\begin{aligned} x + y + z &= 6, \\ 2x - y + 3z &= 14, \\ x + 4y - z &= -2. \end{aligned}$$

3. Determine whether the following system is consistent:

$$\begin{aligned} x - y + 2z &= 3, \\ 2x - 2y + 4z &= 6, \\ 3x - y + z &= 5. \end{aligned}$$

4. Determine whether the following system is inconsistent:

$$\begin{aligned} x + y - z &= 1, \\ 2x + 2y - 2z &= 3. \end{aligned}$$

5. Find all solutions of

$$\begin{aligned} x + y + z &= 2, \\ 2x + 2y + 2z &= 4. \end{aligned}$$

6. Perform Gaussian elimination on the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 5 & -1 & 3 \\ -1 & 1 & 3 & 7 \end{array}\right)$$

   and solve the corresponding system.

7. Explain why the row operation $R_i \leftarrow 0 \cdot R_i$ is not allowed.

8. Give an example of two different linear systems having the same augmented matrix after a sequence of elementary row operations.

Solution.

1. We are given

$$x + 2y = 5, \qquad 3x - y = 4.$$

   From the first equation,

$$x = 5 - 2y.$$

   Substitute into the second equation:

$$3(5 - 2y) - y = 4$$

$$15 - 6y - y = 4$$

$$15 - 7y = 4$$

$$y = \frac{11}{7}.$$

   Substitute back:

$$x = 5 - 2\left(\frac{11}{7}\right) = \frac{35}{7} - \frac{22}{7} = \frac{13}{7}.$$

   Thus the unique solution is

$$\boxed{(x, y) = \left(\frac{13}{7}, \frac{11}{7}\right)}.$$

2. We are given

$$x + y + z = 6,$$
$$2x - y + 3z = 14,$$
$$x + 4y - z = -2.$$

The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 2 & -1 & 3 & | & 14 \\ 1 & 4 & -1 & | & -2 \end{pmatrix}.$$

Perform row operations.

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 - R_1$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -3 & 1 & | & 2 \\ 0 & 3 & -2 & | & -8 \end{pmatrix}.$$

$$R_3 \leftarrow R_3 + R_2$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -3 & 1 & | & 2 \\ 0 & 0 & -1 & | & -6 \end{pmatrix}.$$

From the third row,

$$z = 6.$$

From the second row,

$$-3y + z = 2$$

$$-3y + 6 = 2$$

$$y = \frac{4}{3}.$$

From the first row,

$$x + y + z = 6$$

$$x + \frac{4}{3} + 6 = 6$$

$$x = -\frac{4}{3}.$$

Thus the solution is

$$\boxed{(x, y, z) = \left(-\frac{4}{3}, \frac{4}{3}, 6\right)}.$$

3. In order to determine the consistency of the system

$$x - y + 2z = 3,$$
$$2x - 2y + 4z = 6,$$
$$3x - y + z = 5,$$

we form the augmented matrix:

$$\begin{pmatrix} 1 & -1 & 2 & | & 3 \\ 2 & -2 & 4 & | & 6 \\ 3 & -1 & 1 & | & 5 \end{pmatrix}.$$

We carry out the following row operations:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 - 3R_1$$

$$\begin{pmatrix} 1 & -1 & 2 & | & 3 \\ 0 & 0 & 0 & | & 0 \\ 0 & 2 & -5 & | & -4 \end{pmatrix}.$$

Swap R_2 and R_3:

$$\begin{pmatrix} 1 & -1 & 2 & | & 3 \\ 0 & 2 & -5 & | & -4 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Since there is no contradiction, the system is consistent and has infinitely many solutions. Let $z = t$. From row 2:

$$2y - 5z = -4$$

$$y = \frac{-4 + 5t}{2}.$$

From row 1:

$$x - y + 2z = 3$$

$$x = 3 + y - 2z.$$

Substitute y:

$$x = 3 + \frac{-4 + 5t}{2} - 2t = \frac{2 + t}{2}.$$

Thus the solution is

$$(x, y, z) = \left(\frac{2 + t}{2}, \frac{-4 + 5t}{2}, t \right).$$

Because the system has infinitely many solutions, it is consistent.

4. To determine if the system

$$x + y - z = 1,$$

$$2x + 2y - 2z = 3.$$

is inconsistent, we need to construct the augmented matrix:

$$\begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 2 & 2 & -2 & | & 3 \end{pmatrix}.$$

We will carry out the following row operations:

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}.$$

The second row implies

$$0 = 1,$$

which is impossible.

Therefore the system is inconsistent and has no solution.

5. We are given

$$x + y + z = 2,$$

$$2x + 2y + 2z = 4.$$

Augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 2 & 2 & 2 & | & 4 \end{pmatrix}.$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Thus there are infinitely many solutions.

Let $y = s,\ z = t$.

Then substituting gives

$$x = 2 - s - t.$$

Solution set:

$$\boxed{(x, y, z) = (2 - s - t, s, t), \quad s, t \in \mathbb{R}}.$$

6. To perform Gaussian elimination on

$$\begin{pmatrix} 1 & 2 & 1 & | & 4 \\ 2 & 5 & -1 & | & 3 \\ -1 & 1 & 3 & | & 7 \end{pmatrix},$$

we need to apply a series of row operations:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 + R_1$$

$$\begin{pmatrix} 1 & 2 & 1 & | & 4 \\ 0 & 1 & -3 & | & -5 \\ 0 & 3 & 4 & | & 11 \end{pmatrix}.$$

$$R_3 \leftarrow R_3 - 3R_2$$

$$\begin{pmatrix} 1 & 2 & 1 & | & 4 \\ 0 & 1 & -3 & | & -5 \\ 0 & 0 & 13 & | & 26 \end{pmatrix}.$$

From the third row:

$$z = 2.$$

From the second row:

$$y - 3z = -5$$

$$y = 1.$$

And finally, from the first row:

$$x + 2y + z = 4$$

$$x = 0.$$

Thus the solution to the system is

$$\boxed{(x, y, z) = (0, 1, 2).}$$

7. The row operation $R_i \leftarrow 0 \cdot R_i$ replaces the entire row with zeros.

This operation destroys information contained in the equation represented by that row. Since elementary row operations must preserve the solution set of the system, multiplying a row by zero is not allowed because it changes the system and may introduce additional solutions.

8. Consider the two linear systems:

   System 1:

$$x + y = 2,$$
$$2x + 2y = 4.$$

   System 2:

$$x + y = 2,$$
$$3x + 3y = 6.$$

   The augmented matrices are

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 2 & 2 & | & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & | & 2 \\ 3 & 3 & | & 6 \end{pmatrix}.$$

   Row reducing both gives

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{pmatrix}.$$

   Thus two different systems can lead to the same reduced augmented matrix after elementary row operations.

2.3 Reduced Row-Echelon Matrices

In the previous section, we studied Gaussian elimination as a systematic method for solving linear systems. There, the main objective was to simplify an augmented matrix by using elementary row operations until the corresponding system could be solved by back-substitution. In the course of that procedure, certain special matrix forms arose naturally. These forms are important not merely for computation, but because they reveal the structural features of a linear system in a particularly transparent way.

The present section is devoted to two such forms: *row-echelon form* and *reduced row-echelon form*. A matrix in row-echelon form is already sufficiently organized to permit back-substitution, while a matrix in reduced row-echelon form is even more refined: its solutions may often be read directly from the matrix without further work. These forms play a central role throughout linear algebra, both theoretically and computationally.

Row-Echelon Form We begin by isolating the main structural features produced by Gaussian elimination.

Definition 2.3.1. *A matrix is said to be in* row-echelon form *if it satisfies the following conditions:*

1. *All rows consisting entirely of zeros, if any, occur below all nonzero rows.*
2. *In each nonzero row, the first nonzero entry is* 1*. This entry is called a* leading 1 *or a* pivot.
3. *If one row lies below another nonzero row, then the leading* 1 *in the lower row occurs to the right of the leading* 1 *in the higher row.*

4. *All entries below a leading* 1 *are zero.*

Thus, in row-echelon form, the pivots descend from left to right as one moves downward through the matrix, and each pivot has zeros beneath it.

Example 2.3.1. *The matrix*

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

is in row-echelon form.

Example 2.3.2. *The matrix*

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

is also in row-echelon form, since the zero row appears at the bottom and the pivots move to the right as we descend.

Example 2.3.3. *The matrix*

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

is not *in row-echelon form, because the leading entry of the third row occurs to the left of the leading entry in the second row.*

Reduced Row-Echelon Form Although row-echelon form is already useful, it still requires back-substitution in order to solve a system. A more refined form eliminates even this final step.

Definition 2.3.2. *A matrix is in* reduced row-echelon form *if*

1. *it is in row-echelon form, and*

2. *each leading* 1 *is the only nonzero entry in its column.*

Thus, a reduced row-echelon matrix has all the structural features of row-echelon form, with the additional condition that every pivot has zeros both below and above it.

Example 2.3.4. *The matrix*

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in reduced row-echelon form.

Example 2.3.5. *The matrix*

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

is in row-echelon form but not in reduced row-echelon form, since the pivot in the second column has a nonzero entry above it.

Remark. Every reduced row-echelon matrix is automatically in row-echelon form, but not every row-echelon matrix is reduced.

From Row-Echelon Form to Reduced Row-Echelon Form Gaussian elimination produces row-echelon form. To obtain reduced row-echelon form, one continues the elimination process by clearing the entries *above* each pivot as well. This extended process is often called *Gauss–Jordan elimination.*

Example 2.3.6. *Reduce the matrix*

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

to reduced row-echelon form.

Solution. The matrix is already in row-echelon form. We now eliminate the entries above the pivots.

First clear the entries above the pivot in the third column:

$$R_2 \leftarrow R_2 - 4R_3, \qquad R_1 \leftarrow R_1 + R_3.$$

This gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

Next clear the entry above the pivot in the second column with the following operation:

$$R_1 \leftarrow R_1 - 2R_2.$$

Hence we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 52 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

This is the reduced row-echelon form.

The reduced matrix immediately tells us that the corresponding system has the unique solution

$$x_1 = 52, \qquad x_2 = -22, \qquad x_3 = 5.$$

Problem 2.3.1. *Convert the following matrix to reduced row-echelon form:*

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

and find the solution to the corresponding linear system.

Solution. We convert the matrix to reduced row–echelon form using elementary row operations.

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 5 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

This matrix is already in row–echelon form since:

- Each leading entry (pivot) is to the right of the pivot in the row above.
- All entries below each pivot are zero.

To obtain **reduced** row–echelon form, we must also make all entries *above* each pivot equal to zero.

Step 1: Eliminate the entry above the pivot in column 3.
The pivot in column 3 is in row 3. We eliminate the entry in row 2 column 3 using

$$R_2 \leftarrow R_2 - 2R_3$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

Now eliminate the entry in row 1 column 3 using

$$R_1 \leftarrow R_1 - R_3$$

$$\begin{pmatrix} 1 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

Step 2: Eliminate the entry above the pivot in column 2.
The pivot in column 2 is in row 2. We eliminate the entry in row 1 column 2 using

$$R_1 \leftarrow R_1 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

Now every pivot is equal to 1, and all other entries in each pivot column are zero. Therefore the matrix is in **reduced row–echelon form**.

$$\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

This corresponds to the system of equations

$$x = 3, \qquad y = 1, \qquad z = 2.$$

Thus the solution to the system is

$$\boxed{(x, y, z) = (3, 1, 2)}.$$

Reading Solutions from Reduced Row-Echelon Form One of the chief advantages of reduced row-echelon form is that the solution set of a system becomes especially easy to interpret.

Example 2.3.7 (A unique solution)**.** *Consider the augmented matrix*

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right).$$

The corresponding system is simply

$$x_1 = 3, \qquad x_2 = -1, \qquad x_3 = 4.$$

Thus the system has a unique solution.

Example 2.3.8 (Infinitely many solutions)**.** *Consider the matrix*

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 5 \\ 0 & 1 & -3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The corresponding system is

$$x_1 + 2x_3 - x_4 = 5,$$
$$x_2 - 3x_3 + 4x_4 = 2.$$

Here x_3 and x_4 are not pivot variables, so they are free variables. Let

$$x_3 = s, \qquad x_4 = t,$$

where $s, t \in \mathbb{R}$. Then

$$x_1 = 5 - 2s + t, \qquad x_2 = 2 + 3s - 4t.$$

Hence the solution set is

$$(x_1, x_2, x_3, x_4) = (5 - 2s + t,\ 2 + 3s - 4t,\ s,\ t), \qquad s, t \in \mathbb{R}.$$

Example 2.3.9 (No solution)**.** *Consider*

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The third row corresponds to the equation

$$0 = 1,$$

which is impossible. Therefore the system is inconsistent and has no solution.

These examples show that reduced row-echelon form reveals the three possibilities for a linear system very clearly:

1. a unique solution,
2. infinitely many solutions,
3. no solution.

Pivot Columns and Free Variables The structure of a reduced row-echelon matrix naturally divides the variables into two types.

Definition 2.3.3. *A variable corresponding to a pivot column is called a* basic variable. *A variable corresponding to a non-pivot column is called a* free variable.

Basic variables are determined by the equations, while free variables may be assigned arbitrary values.

Example 2.3.10. *Consider the matrix*

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 7 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

The pivot columns are columns 1, 2, *and* 4. *Hence* x_1, x_2, *and* x_4 *are basic variables, while* x_3 *is free.*

Let $x_3 = t$. *Then*

$$x_1 = 7 - 3t, \qquad x_2 = -1 + 2t, \qquad x_4 = 4.$$

So the solution set is

$$(x_1, x_2, x_3, x_4) = (7 - 3t,\ -1 + 2t,\ t,\ 4), \qquad t \in \mathbb{R}.$$

Problem 2.3.2. *For each of the following reduced row-echelon matrices, identify the basic and free variables, and describe the solution set:*

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & 1 \end{array}\right), \qquad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -2 \end{array}\right).$$

Uniqueness of Reduced Row-Echelon Form A remarkable fact about reduced row-echelon form is that it is uniquely determined by the matrix.

Theorem 2.3.1. *Every matrix is row-equivalent to exactly one reduced row-echelon matrix.*

This theorem is extremely important. Although there may be many different sequences of elementary row operations that can be applied to a matrix, all such valid sequences lead ultimately to the same reduced row-echelon form.

Remark. Row-echelon form need not be unique. Two different sequences of row operations may lead to different row-echelon matrices. However, once reduction is carried all the way to reduced row-echelon form, the result is unique.

Example 2.3.11. *Starting with the matrix*

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

one might first perform

$$R_2 \leftarrow R_2 - 2R_1$$

to obtain

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix},$$

which is already in row-echelon form. Another person might first interchange rows or scale a row before eliminating. The intermediate matrices may differ, but the reduced row-echelon form will always be

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

A Complete Worked Example We now illustrate the entire process on a system that has infinitely many solutions.

Example 2.3.12. *Solve the system*

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 3, \\ 2x_1 + 4x_2 + x_3 - x_4 &= 7, \\ -x_1 - 2x_2 + 2x_3 &= -1. \end{aligned}$$

Solution. The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 2 & 4 & 1 & -1 & 7 \\ -1 & -2 & 2 & 0 & -1 \end{array}\right).$$

First eliminate the entries below the pivot in the first column:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 + R_1.$$

This yields

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right).$$

Interchange the second and third rows:

$$R_2 \leftrightarrow R_3,$$

so that

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 & 1 \end{array}\right).$$

Now eliminate below the pivot in the third column:

$$R_3 \leftarrow R_3 - 3R_2,$$

giving

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -6 & -5 \end{array}\right).$$

Scale the third row:

$$R_3 \leftarrow -\frac{1}{6}R_3,$$

to obtain

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & \frac{5}{6} \end{array}\right).$$

Now clear above the pivot in column four:

$$R_1 \leftarrow R_1 - R_3, \qquad R_2 \leftarrow R_2 - R_3.$$

This gives

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & \frac{13}{6} \\ 0 & 0 & 1 & 0 & \frac{7}{6} \\ 0 & 0 & 0 & 1 & \frac{5}{6} \end{array}\right).$$

Finally clear above the pivot in column three:

$$R_1 \leftarrow R_1 + R_2,$$

and obtain

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & \frac{10}{3} \\ 0 & 0 & 1 & 0 & \frac{7}{6} \\ 0 & 0 & 0 & 1 & \frac{5}{6} \end{array}\right).$$

This is the reduced row-echelon form.

The pivot columns are 1, 3, and 4, so x_1, x_3, and x_4 are basic variables, while x_2 is free. Let

$$x_2 = t.$$

Then from the reduced matrix we have

$$x_1 + 2x_2 = \frac{10}{3}, \qquad x_3 = \frac{7}{6}, \qquad x_4 = \frac{5}{6}.$$

Hence

$$x_1 = \frac{10}{3} - 2t.$$

Therefore the solution set is

$$(x_1, x_2, x_3, x_4) = \left(\frac{10}{3} - 2t,\ t,\ \frac{7}{6},\ \frac{5}{6}\right), \qquad t \in \mathbb{R}.$$

Exercises

1. Reduce the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

to reduced row-echelon form.

2. Reduce the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

to reduced row-echelon form and solve the corresponding system.

3. Solve the system
$$\begin{aligned} x + y + z &= 2, \\ 2x + 2y + 2z &= 4, \\ x - y + z &= 0. \end{aligned}$$

4. Determine the basic and free variables for the system whose reduced row-echelon form is
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 4 \\ 0 & 1 & -2 & 5 & 7 \end{array}\right).$$

5. Give an example of a matrix that is in row-echelon form but not in reduced row-echelon form.

6. Give an example of a system of linear equations whose reduced row-echelon form contains a row of the form
$$(\, 0 \;\; 0 \;\; 0 \mid 1 \,).$$

7. Explain why a system with more variables than pivot columns must have at least one free variable.

8. Prove that the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
is in reduced row-echelon form.

Solution.

1. We have an upper triangular matrix, so the pivots are already in place. Our goal is to clear the entries above each pivot to reach reduced row-echelon form.

   First, we eliminate the entries above the pivot in the third column:
$$R_2 \leftarrow R_2 - 3R_3, \quad R_1 \leftarrow R_1 - R_3 \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Next, we eliminate the entry above the pivot in the second column:
$$R_1 \leftarrow R_1 - 2R_2 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   At this point, each pivot is the only nonzero entry in its column, so the matrix is in reduced row-echelon form.

2. We again start with a matrix that is already close to row-echelon form. The strategy is the same: work from the bottom pivot upward and clear entries above each pivot.

   First, eliminate the entries above the pivot in the third column:
$$R_2 \leftarrow R_2 - 2R_3, \quad R_1 \leftarrow R_1 - R_3 \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Then eliminate the entry above the pivot in the second column:

$$R_1 \leftarrow R_1 - R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Now the matrix is in reduced row-echelon form. Reading off the solution directly from the last column, we obtain

$$\boxed{(x, y, z) = (1, 1, 2).}$$

3. We notice immediately that the second equation is just a multiple of the first, so it does not provide new information. This suggests the system may have infinitely many solutions.

   To proceed, we combine the first and third equations. Subtracting the third from the first gives
   $$2y = 2 \Rightarrow y = 1.$$
   Substituting this back into one of the original equations, say $x - y + z = 0$, we get
   $$x - 1 + z = 0 \Rightarrow x + z = 1.$$
   Since this is a single equation in two variables, we let one variable be free. Let $z = t$. Then $x = 1 - t$, and the solution set can be written as
   $$\boxed{(x, y, z) = (1 - t, 1, t), \quad t \in \mathbb{R}.}$$

4. We interpret the reduced row-echelon form by identifying pivot columns. The leading 1s appear in the first and second columns, so those correspond to basic variables.

   This means x_1 and x_2 are determined by the system, while the remaining variables do not have pivots and can vary freely. Hence x_3 and x_4 are free variables.

5. To construct such an example, we want a matrix that satisfies the conditions of row-echelon form (pivots stepping to the right, zeros below pivots), but fails the stricter reduced conditions.

   For instance,
   $$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$
   is in row-echelon form, since each pivot moves to the right and entries below pivots are zero. However, it is not reduced because there are nonzero entries above the pivots.

6. We want a system that becomes inconsistent after row reduction. A simple way to achieve this is to take two equations with identical left-hand sides but different constants:
   $$\begin{cases} x + y = 1, \\ x + y = 2. \end{cases}$$
   Subtracting one equation from the other yields $0 = 1$, which corresponds to a row of the form $(0\ 0\ 0 \mid 1)$ in reduced row-echelon form.

7. We recall that pivot columns correspond to variables that are solved for (basic variables). If there are more variables than pivot columns, then some variables cannot be leading variables. Those variables without pivots are, by definition, free variables. Therefore, having more variables than pivot columns guarantees the existence of at least one free variable.

8. Finally, we verify directly that the given matrix satisfies all the conditions for reduced row-echelon form. Each row has a leading 1, these leading 1s move to the right as we go down the rows, and each pivot column contains zeros everywhere else. Since all defining properties are satisfied, the matrix is indeed in reduced row-echelon form.

2.4 Matrix Operations

Having established the use of matrices for representing linear systems and understanding row-echelon and reduced row-echelon forms, we now focus on *operations* on matrices themselves. Matrix operations extend the utility of matrices beyond solving systems: they allow us to manipulate, combine, and transform matrices algebraically. These operations form the foundation for subsequent topics, including inverses, determinants, and linear transformations.

Addition and Scalar Multiplication

Definition 2.4.1 (Matrix Addition)**.** *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the sum $A + B$ is the $m \times n$ matrix defined by*

$$(A + B)_{ij} = a_{ij} + b_{ij}, \quad 1 \le i \le m,\ 1 \le j \le n.$$

Example 2.4.1. *If*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}.$$

Problem 2.4.1. *Compute the sum of*

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 & 0 \\ 3 & -2 & 2 \end{pmatrix}.$$

Solution. We simply add up individual entries to produce a resulting matrix of the same size as the two addends:

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 & 0 \\ 3 & -2 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 4 & 3 \\ 3 & 2 & 3 \end{pmatrix}}$$

Definition 2.4.2 (Scalar Multiplication)**.** *Let $A = [a_{ij}]$ be an $m \times n$ matrix and $k \in \mathbb{R}$. Then the scalar multiple kA is defined by*

$$(kA)_{ij} = ka_{ij}.$$

Example 2.4.2. *If*

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}, \quad k = 4,$$

then

$$4A = \begin{pmatrix} 4 & -8 \\ 12 & 0 \end{pmatrix}.$$

Problem 2.4.2. *Compute* $-3\begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix}$.

Solution. Call the resulting matrix $A = (a_{ij})$. We have $a_{11} = -3(2) = -6$, $a_{12} = -3(0) = 0$, $a_{21} = -3(-1) = 3$, and $a_{22} = -3(4) = -12$. So the answer is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \boxed{\begin{pmatrix} -6 & 0 \\ 3 & -12 \end{pmatrix}}$$

Properties of Matrix Addition and Scalar Multiplication

Theorem 2.4.1 (Basic Properties)**.** *Let A, B, C be $m \times n$ matrices, and let $k, \ell \in \mathbb{R}$. Then the following eight properties hold:*

1. $A + B = B + A$ *(Commutativity)*
2. $(A + B) + C = A + (B + C)$ *(Associativity)*
3. $A + 0 = A$*, where* 0 *is the* $m \times n$ *zero matrix*
4. $A + (-A) = 0$
5. $k(A + B) = kA + kB$ *(Distributivity over matrix addition)*
6. $(k + \ell)A = kA + \ell A$ *(Distributivity over scalars)*
7. $(k\ell)A = k(\ell A)$
8. $1 \cdot A = A$

Problem 2.4.3. *Verify the distributive property* $2(A + B) = 2A + 2B$ *for*

$$A = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}.$$

Solution.

$$2(A + B) = 2A + 2B.$$

Step 1: Compute $A + B$**.** We add the two matrices entry-wise:

$$A + B = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 + (-2) & 0 + 1 \\ 3 + 4 & -1 + 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 7 & 1 \end{pmatrix}.$$

Step 2: Compute $2(A + B)$. Multiply each entry by 2:

$$2(A+B) = 2\begin{pmatrix} -1 & 1 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 14 & 2 \end{pmatrix}.$$

Step 3: Compute $2A$ and $2B$ separately.

$$2A = 2\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 6 & -2 \end{pmatrix}.$$

$$2B = 2\begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 8 & 4 \end{pmatrix}.$$

Step 4: Add $2A + 2B$. We add the two scaled matrices:

$$2A + 2B = \begin{pmatrix} 2 & 0 \\ 6 & -2 \end{pmatrix} + \begin{pmatrix} -4 & 2 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 2+(-4) & 0+2 \\ 6+8 & -2+4 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 14 & 2 \end{pmatrix}.$$

We observe that:

$$2(A+B) = \begin{pmatrix} -2 & 2 \\ 14 & 2 \end{pmatrix} \quad \text{and} \quad 2A + 2B = \begin{pmatrix} -2 & 2 \\ 14 & 2 \end{pmatrix}.$$

As you can see, there is a match, and so we have verified the distributive property. □

Matrix Multiplication Matrix multiplication is more subtle than addition or scalar multiplication. Not all matrices can be multiplied, and the order of multiplication matters.

Definition 2.4.3 (Matrix Multiplication)**.** *Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Then the product AB is the $m \times p$ matrix whose (i, j) entry is*

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example 2.4.3. *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

Their product AB is

$$AB = \begin{pmatrix} 1\cdot 0 + 2\cdot 2 & 1\cdot 1 + 2\cdot(-1) \\ 3\cdot 0 + 4\cdot 2 & 3\cdot 1 + 4\cdot(-1) \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 8 & -1 \end{pmatrix}.$$

Problem 2.4.4. *Compute*

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

Solution. To multiply the matrices, we compute the dot products of the rows of the first matrix with the columns of the second matrix:

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 0\cdot 0 + 2\cdot 1 & 1\cdot 1 + 0\cdot(-1) + 2\cdot 2 \\ -1\cdot 2 + 3\cdot 0 + 1\cdot 1 & -1\cdot 1 + 3\cdot(-1) + 1\cdot 2 \end{pmatrix} = \boxed{\begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix}}$$

Remark. Matrix multiplication is *associative* but generally *not commutative*, i.e.,

$$AB \neq BA$$

in general, although there may be some rare exceptions.

The Identity Matrix

Definition 2.4.4 (Identity Matrix)**.** *The $n \times n$ identity matrix I_n is the matrix with 1s on the diagonal and 0s elsewhere:*

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Theorem 2.4.2 (Multiplicative Identity)**.** *For any $m \times n$ matrix A,*

$$AI_n = A, \quad I_m A = A.$$

Example 2.4.4. *If*

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then it can be shown that

$$AI_2 = I_2 A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}.$$

Transpose of a Matrix

Definition 2.4.5 (Transpose)**.** *For an $m \times n$ matrix $A = (a_{ij})$, the transpose A^T is the $n \times m$ matrix defined by*

$$(A^T)_{ij} = a_{ji}.$$

Example 2.4.5. *If*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

then

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Theorem 2.4.3 (Properties of Transpose)**.** *For matrices A and B (of compatible sizes) and scalar k:*

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$

Problem 2.4.5. *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}.$$

Compute $(A + B)^T$ and $A^T + B^T$ to verify the transpose property.

Solution. We first find the sum $A + B$.

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \\ 6 & 1 \end{pmatrix} \end{aligned}$$

The quantity $(A+B)^T$ is the transpose of the above sum:

$$(A+B)^T = \begin{pmatrix} 4 & 1 \\ 6 & 1 \end{pmatrix}^T = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix}$$

Now, we find each individual transposed matrix. We have

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}^T = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

We now add these two matrices:

$$A^T + B^T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix}$$

We see that

$$(A+B)^T = A^T + B^T = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix}$$

and so we have verified the transpose property. □

Exercises

1. Compute $A + B$ and $2A - 3B$ for

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}.$$

2. Verify that $(kA)^T = kA^T$ for

$$k = -2, \quad A = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}.$$

3. Multiply

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}.$$

4. Find the transpose of

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \end{pmatrix}.$$

5. Show by example that $AB \neq BA$ for 2×2 matrices.

6. Verify that $AI = IA = A$ for $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$.

7. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Verify that $(AB)^T = B^T A^T$.

8. If A is 3×2 and B is 2×3, compute the size of AB and BA. Can they be equal?

9. Compute $3A - 2B + 5I_2$ for

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

10. Prove that $(A^T)^T = A$ for a 2×3 matrix A.

Solution.

1. We are given

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}.$$

First, we find the sum $A + B$:

$$A + B = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1+4 & -2+1 \\ 0+(-1) & 3+2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}.$$

Next, perform scalar multiplication on A and B:

$$2A = 2\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 0 & 6 \end{pmatrix}, \qquad 3B = 3\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ -3 & 6 \end{pmatrix}.$$

Thus, the answer is

$$2A - 3B = \begin{pmatrix} 2 & -4 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 12 & 3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 2-12 & -4-3 \\ 0-(-3) & 6-6 \end{pmatrix} = \boxed{\begin{pmatrix} -10 & -7 \\ 3 & 0 \end{pmatrix}.}$$

2. We are given

$$k = -2, \qquad A = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}.$$

First compute kA:

$$kA = -2\begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ -2 & 2 \end{pmatrix}.$$

Then

$$(kA)^T = \begin{pmatrix} -6 & 0 \\ -2 & 2 \end{pmatrix}^T = \begin{pmatrix} -6 & -2 \\ 0 & 2 \end{pmatrix}.$$

Now we determine the matrix kA^T. Since

$$A^T = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix},$$

this yields

$$kA^T = -2\begin{pmatrix}3 & 1\\ 0 & -1\end{pmatrix} = \begin{pmatrix}-6 & -2\\ 0 & 2\end{pmatrix}.$$

Hence we have shown that

$$(kA)^T = kA^T.$$

□

3. We wish to compute the product

$$\begin{pmatrix}2 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 3 & 2\end{pmatrix}.$$

We multiply rows by columns:

$$\begin{pmatrix}2\cdot 1 + 1\cdot 3 & 2\cdot 0 + 1\cdot 2\\ 0\cdot 1 + (-1)\cdot 3 & 0\cdot 0 + (-1)\cdot 2\end{pmatrix} = \boxed{\begin{pmatrix}5 & 2\\ -3 & -2\end{pmatrix}.}$$

4. We are asked to find the transpose of

$$\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & 4\end{pmatrix}.$$

Interchanging rows and columns gives

$$\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & 4\end{pmatrix}^T = \boxed{\begin{pmatrix}1 & 0\\ 2 & -1\\ 3 & 4\end{pmatrix}.}$$

5. To show by example that $AB \neq BA$, choose

$$A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \qquad B = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

The first product is

$$AB = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix},$$

while the other is

$$BA = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}.$$

Since

$$\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \neq \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix},$$

this proves by example that $AB \neq BA$ in general. □

6. We are given

$$A = \begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}, \qquad I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

First,

$$AI = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 + (-2) \cdot 0 & 3 \cdot 0 + (-2) \cdot 1 \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = A.$$

Also,

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 0 \cdot 1 & 1 \cdot (-2) + 0 \cdot 0 \\ 0 \cdot 3 + 1 \cdot 1 & 0 \cdot (-2) + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = A.$$

Therefore, we have shown

$$AI = IA = A.$$

□

7. We are given

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

First compute AB:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + 2(-1) & 1 \cdot 1 + 2 \cdot 2 \\ 3 \cdot 0 + 4(-1) & 3 \cdot 1 + 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ -4 & 11 \end{pmatrix}.$$

Hence

$$(AB)^T = \begin{pmatrix} -2 & 5 \\ -4 & 11 \end{pmatrix}^T = \begin{pmatrix} -2 & -4 \\ 5 & 11 \end{pmatrix}.$$

Now compute $B^T A^T$. We have

$$B^T = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \qquad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Therefore,

$$B^T A^T = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + (-1) \cdot 2 & 0 \cdot 3 + (-1) \cdot 4 \\ 1 \cdot 1 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 5 & 11 \end{pmatrix}.$$

Thus, we have demonstrated

$$(AB)^T = B^T A^T.$$

□

8. If A is 3×2 and B is 2×3, then:
   - AB is defined because the inner dimensions match $(2 = 2)$, and its size is

     $$(3 \times 2)(2 \times 3) = 3 \times 3.$$

   - BA is also defined because the inner dimensions match $(3 = 3)$, and its overall size/dimension is

     $$(2 \times 3)(3 \times 2) = 2 \times 2.$$

So we know

$$AB \text{ is } 3 \times 3, \qquad BA \text{ is } 2 \times 2.$$

They cannot be equal, because matrices can only be equal if they have the same size, and these do not. □

9. We are given

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

First compute the scaled matrices:

$$3A = 3\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix},$$

$$2B = 2\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -2 & 6 \end{pmatrix}.$$

Also, the identity matrix scaled by a factor of 5 is

$$5I_2 = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

Therefore,

$$3A - 2B + 5I_2 = \begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ -2 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

First subtract the matrices

$$\begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix}.$$

Then add $5I_2$:

$$\begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & 5 \end{pmatrix}.$$

Hence, the answer is

$$3A - 2B + 5I_2 = \boxed{\begin{pmatrix} 4 & -3 \\ 2 & 5 \end{pmatrix}.}$$

10. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

be any 2×3 matrix. Then its transpose is

$$A^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

Taking the transpose again,

$$(A^T)^T = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

But this is exactly the original matrix A. Therefore,

$$(A^T)^T = A.$$

This proves the statement for any 2×3 matrix A. □

2.5 Inverses of Matrices

In the previous section we introduced matrix multiplication and the identity matrix. Recall that for any $n \times n$ matrix A, multiplication by the identity matrix I_n leaves the matrix unchanged:

$$AI_n = I_n A = A.$$

This observation suggests an important question: can we find a matrix which *undoes* the effect of multiplying by A? In other words, does there exist a matrix B such that

$$AB = BA = I_n?$$

When such a matrix exists, it allows us to solve linear systems directly using matrix multiplication. This idea leads to one of the most important concepts in linear algebra.

Definition of the Inverse

Definition 2.5.1. *Let A be an $n \times n$ matrix. A matrix B is called the **inverse** of A if*

$$AB = BA = I_n.$$

*If such a matrix exists, we say that A is **invertible** (or **nonsingular**). The inverse of A is denoted by A^{-1}.*

Thus an inverse matrix plays a role analogous to the reciprocal of a number. Just as $a \cdot a^{-1} = 1$ for a nonzero real number a, we have

$$AA^{-1} = A^{-1}A = I.$$

Example 2.5.1. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.$$

One can verify that

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Indeed,

$$AA^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Problem 2.5.1. *Verify that*

$$A^{-1} = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}$$

is the inverse of

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}.$$

Solution. Since A is 2×2, we need to check if $AA^{-1} = I_2$. We compute the left hand side:

$$AA^{-1} = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + -1 \cdot 3 & 4 \cdot 1 + -1 \cdot 4 \\ -3 \cdot 1 + 1 \cdot 3 & -3 \cdot 1 + 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We see that

$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Hence, we can verify that the two matrices are inverses. □

Uniqueness of the Inverse An important property of inverses is that they are unique.

Theorem 2.5.1. *If A is invertible, then its inverse is unique.*

Proof. Suppose B and C are both inverses of A. Then

$$AB = BA = I, \qquad AC = CA = I.$$

Observe that

$$B = BI = B(AC) = (BA)C = IC = C.$$

Therefore $B = C$, and the inverse is unique. □

Solving Linear Systems Using Inverses One of the most useful applications of inverses is solving linear systems.

Consider the system represented by the below matrix equation:

$$A\mathbf{x} = \mathbf{b},$$

where A is an $n \times n$ matrix.

If A is invertible, we may multiply both sides by A^{-1}:

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

Since $A^{-1}A = I$, we obtain

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Thus the solution of the system can be written directly in terms of the inverse.

Example 2.5.2. *Solve the system*

$$x + 2y = 5$$
$$3x + 4y = 11.$$

The coefficient matrix is

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}.$$

First compute

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then the unique solution is

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Multiplying the two matrices gives

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(You can verify this for yourself.)
Thus $x = 1$ and $y = 2$ and the solution is the ordered pair $(1, 2)$.

Finding Inverses Using Row Reduction A systematic method for computing inverses uses row operations.

Theorem 2.5.2. *Let A be an $n \times n$ matrix. If the augmented matrix*

$$(A \mid I_n)$$

can be row-reduced to the form

$$(I_n \mid B),$$

then $B = A^{-1}$.

Example 2.5.3. *Find the inverse of*

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

Begin with

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right).$$

Apply row operations:

$$R_1 \leftarrow \tfrac{1}{2}R_1$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 3 & 0 & 1 \end{array}\right)$$

$$R_2 \leftarrow R_2 - 5R_1$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right)$$

$$R_2 \leftarrow 2R_2$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{array}\right)$$

$$R_1 \leftarrow R_1 - \tfrac{1}{2}R_2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right).$$

Thus the inverse matrix is

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Properties of Inverse Matrices

Theorem 2.5.3. *If A and B are invertible matrices of the same size, then*

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

Example 2.5.4. *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

One may verify directly using matrix multiplication that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Non-Invertible Matrices Not every matrix has an inverse.

Example 2.5.5. *Consider*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Performing row reduction yields

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

Since this matrix cannot be reduced to the identity matrix, A is not invertible.

Matrices that do not possess inverses are called **singular matrices**.

Problem 2.5.2. *Determine whether the following matrix is invertible:*

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}.$$

Solution. We perform elementary row operations and see if we can reduce it to the identity matrix:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We recognize that the second row is 2 multiplied by the first, so applying the row operation $R_2 \leftarrow -2R_1 + R_2$ to yield

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}.$$

Since we cannot reduce this matrix further, and we have not obtained the identity matrix, the matrix is not invertible. □

Exercises

1. Determine whether the matrix
$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$
is invertible.

2. Show that $(A^T)^{-1} = (A^{-1})^T$ for an invertible matrix.

3. Find the inverse of
$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

1. The second row of the matrix is 2 times the first. Therefore, we can carry out the following row operation:
$$R_2 \leftarrow -2R_1 + R_2$$
this results in the matrix
$$\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}.$$
Since we cannot reduce this matrix any further, and we have not obtained the identity matrix I_2, the matrix is not invertible. □

2. Using $(AB)^T = B^T A^T$ and $AA^{-1} = I$, observe:
$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I,$$
and similarly
$$(A^{-1})^T A^T = (AA^{-1})^T = I.$$
Thus $(A^{-1})^T$ is the inverse of A^T, so
$$(A^T)^{-1} = (A^{-1})^T.$$
□

3. The matrix is upper triangular with ones on the diagonal. Let
$$A^{-1} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$
Solving $AA^{-1} = I$ gives $a = d = f = 1$, $b = 0$, $e = 1$, and $c = -2$. Hence the inverse matrix is
$$\boxed{A^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

2.6 Determinants

In the previous section, we introduced the concept of an inverse matrix and saw that not every square matrix possesses an inverse. This raises a natural question: how can we determine efficiently whether a matrix is invertible? The answer lies in an important scalar quantity associated with square matrices known as the *determinant*.

Determinants provide a convenient test for invertibility, appear naturally in formulas for matrix inverses, and have important geometric interpretations related to area and volume. In this section we introduce determinants, develop methods for computing them, and explore their fundamental properties.

Determinants of 2×2 Matrices We begin with the simplest case.

Definition 2.6.1. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*The **determinant** of A is defined as*

$$\det(A) = ad - bc.$$

The determinant of A is often denoted by

$$\det(A) \quad \text{or} \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Example 2.6.1. *Compute the determinant of*

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}.$$

$$\det(A) = (3)(4) - (2)(5) = 12 - 10 = 2.$$

Problem 2.6.1. *Compute the determinant of the following matrices.*

$$\begin{pmatrix} 4 & 1 \\ 7 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & -5 \\ 6 & 1 \end{pmatrix}.$$

Solution. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $|A| = ad - bc$. So the requested determinants are

$$\det \begin{pmatrix} 4 & 1 \\ 7 & 3 \end{pmatrix} = 4(3) - 7(1) = 12 - 7 = \boxed{5}$$

$$\det \begin{pmatrix} 2 & -5 \\ 6 & 1 \end{pmatrix} = 2(1) - 6(-5) = 2 + 30 = \boxed{32}$$

Determinants and Invertibility Determinants immediately provide a criterion for invertibility of matrices.

Theorem 2.6.1. *A 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if

$$\det(A) \neq 0.$$

Example 2.6.2. *Consider*

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

$$\det(A) = 2(2) - 4(1) = 4 - 4 = 0.$$

Thus A is not invertible.

Problem 2.6.2. *Determine whether the matrix is invertible.*

$$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

Solution. We need to find the determinant of this matrix. Consider

$$\det \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = 3(2) - 4(1) = 6 - 4 = 2 \neq 0$$

Since the determinant is nonzero, the matrix is invertible. □

Determinants of 3×3 Matrices For larger matrices, determinants are computed using **cofactor expansion**.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The determinant of A is

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 2.6.3. *Compute*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

$$\det(A) = 1 \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} + 3 \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix}$$

$$= 1(24) - 2(-5) + 3(-4)$$

$$= 24 + 10 - 12 = 22.$$

Problem 2.6.3. *Compute*

$$\det\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 4 \\ 5 & 2 & 0 \end{pmatrix}.$$

Solution. We proceed using cofactor expansion (along the first row). So

$$\det = 2\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 2(0-8) + 1(6-5) = 2(-8) + 1(1) = \boxed{-15}$$

Minors and Cofactors To define determinants for larger matrices we introduce minors and cofactors.

Definition 2.6.2. *Let A be an $n \times n$ matrix.*
The minor M_{ij} *is the determinant obtained by deleting row i and column j.*
The cofactor C_{ij} *is*

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Thus determinant expansion can be written

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

This is called **expansion along the first row**.

Problem 2.6.4. *Find the minor and cofactor of entry a_{23} for*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

Solution. The minor M_{23} is the determinant obtained by deleting the second row and third column. So we have

$$M_{23} = \begin{vmatrix} 1 & 0 \\ 6 & 7 \end{vmatrix} = 1(7) - 6(0) = \boxed{7}$$

The cofactor is simply the minor multiplied by $(-1)^{i+j}$, for the i-th row and j-th column. So we have

$$C_{23} = (-1)^{2+3} M_{23} = -1 \cdot 7 = \boxed{-7}$$

Properties of Determinants Determinants satisfy several important properties.

Theorem 2.6.2. *Let A be an $n \times n$ matrix.*

1. *If two rows of A are equal then* $\det(A) = 0$.
2. *Swapping two rows multiplies the determinant by -1.*
3. *Multiplying a row by a constant k multiplies the determinant by k.*

4. *If a row is a linear combination of other rows then* $\det(A) = 0$.

5. $\det(AB) = \det(A)\det(B)$.

Example 2.6.4. *Compute*

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

Since the second row is twice the first row, the determinant is

$$\det(A) = 0.$$

Problem 2.6.5. *Explain why the determinant is zero.*

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 12 & 8 \\ 2 & 6 & 4 \end{pmatrix}.$$

Solution. *Call* R_i *the* i*-th row of the matrix. Here, we take* $i = 1, 2, 3$. *It is clear that* $R_2 = 4R_1$ *and* $R_3 = 2R_1$. *Since all the rows are linear combinations of each other, the determinant of the matrix is zero.* □

Determinants and Row Reduction Row operations affect determinants in some intuitive ways.

- Swapping rows changes the sign of the determinant.
- Multiplying a row by k multiplies the determinant by k.
- Adding a multiple of one row to another does not change the determinant.

These properties allow determinants to be computed efficiently using row reduction.

Example 2.6.5. *Compute*

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Row reduction produces

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}.$$

The third row is a multiple of the second, so the determinant is

$$\det(A) = 0.$$

Determinants and Inverses Determinants give a complete test for invertibility.

Theorem 2.6.3. *A square matrix A is invertible if and only if*

$$\det(A) \neq 0.$$

Thus determinants provide a simple test for non-singularity.

Geometric Interpretation It's not necessary for this course, but it's important to keep in mind that determinants also serve a geometric purpose. Specifically:

- For 2×2 matrices, $|\det(A)|$ equals the **area scaling factor**.
- For 3×3 matrices, $|\det(A)|$ equals the **volume scaling factor**.

Thus determinants measure how linear transformations stretch or shrink space.

Exercises

1. Compute the determinant using row reduction:

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \\ 2 & 5 & 3 \end{pmatrix}.$$

2. Show that if a matrix has two identical rows then its determinant is zero.
3. Prove that if $\det(A) = 0$, then A cannot have an inverse.
4. Find the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}.$$

Solution.

1. Perform row reduction:

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \\ 2 & 5 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 4R_1,\ R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -12 & -7 \\ 0 & -1 & -1 \end{pmatrix}.$$

Next eliminate:

$$R_3 \to R_3 - \tfrac{1}{12} R_2 \Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & -12 & -7 \\ 0 & 0 & -\frac{5}{12} \end{pmatrix}.$$

Since only row additions were used, the determinant is the product along the diagonal:

$$\det = 1 \cdot (-12) \cdot \left(-\tfrac{5}{12}\right) = \boxed{5}.$$

2. Suppose a matrix has two identical rows, say $R_i = R_j$ with $i \neq j$. Swap these two rows. The determinant changes sign, but the matrix is unchanged, so

$$\det(A) = -\det(A).$$

For this statement be true, we have $\det(A) = 0$. $\square$

3. If $\det(A) = 0$, then we know A is singular (not invertible). Suppose (for proof by contradiction) that A is invertible. Then we obtain

$$\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = 0,$$

which is impossible, since $\det(I) = 1$ (taking the product along the main diagonal). Therefore A cannot have an inverse. $\square$

4. We apply cofactor expansion along the first row:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}.$$

Compute each minor:

$$= 1(50-48) - 2(40-42) + 3(32-35) = 2 - 2(-2) + 3(-3) = 2 + 4 - 9 = \boxed{-3}.$$

3 Vector Spaces

Vector spaces form the foundational structure of linear algebra. Although vectors are often first encountered as geometric objects in the plane or in three-dimensional space, the concept extends far beyond geometry. By abstracting the essential properties of linear combination and scalar multiplication, vector spaces provide a unified framework for studying linear equations, transformations, and coordinate systems. The ideas developed in this chapter—subspaces, span, linear independence, basis, and dimension—establish the structural language that underlies the rest of linear algebra.

3.1 The Vector Space $\mathbb{R}^3$

In this section, we introduce vectors in three-dimensional space, $\mathbb{R}^3$. You have probably seen vectors in physics or geometry as arrows in space, representing quantities like force, velocity, or displacement. Here, we take that geometric idea and connect it to the algebraic structure of a *vector space.*

Our goals are to:

- Understand vectors in $\mathbb{R}^3$ as ordered triples and as arrows in space.
- Define vector addition and scalar multiplication and see their geometric meaning.
- Verify basic vector identities and practice computations.
- Introduce the dot product as a way to measure length and angles.

Vectors in $\mathbb{R}^3$ as ordered triples.

Definition 3.1.1. *The set $\mathbb{R}^3$ consists of all ordered triples of real numbers:*

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

An element $(x, y, z) \in \mathbb{R}^3$ is called a vector *in $\mathbb{R}^3$.*

We often write vectors as column vectors,

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

or as boldface letters like $\mathbf{v}$, when the coordinates are understood.

Geometric picture: arrows from the origin. You can think of a vector $\mathbf{v} = (x, y, z)$ as an arrow in three-dimensional space:

- The tail of the arrow is at the origin $(0, 0, 0)$.
- The head of the arrow is at the point (x, y, z).

Even though we draw vectors as arrows, we really care about the *displacement* they represent, not the specific location of the arrow. So any arrow with the same direction and length is considered the same vector.

Vector addition and scalar multiplication.

Definition 3.1.2. *Given vectors* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *in* $\mathbb{R}^3$*, and a real number* c *(called a* scalar*), we define:*

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3),$$

$$c\mathbf{v} = (cv_1,\ cv_2,\ cv_3).$$

Geometrically:

- Vector addition corresponds to placing the tail of $\mathbf{v}$ at the head of $\mathbf{u}$ and drawing the arrow from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$ (the "tip-to-tail" rule).
- Scalar multiplication by $c > 0$ stretches or shrinks the arrow by a factor of $|c|$; if $c < 0$, it also reverses direction.

Example 3.1.1: Computing vector operations. Let

$$\mathbf{u} = (1, -2, 3), \quad \mathbf{v} = (4, 0, -1), \quad c = -2.$$

Compute the following vector operations:

$$\mathbf{u} + \mathbf{v}, \quad \mathbf{u} - \mathbf{v}, \quad c\mathbf{v}.$$

Solution. We compute component-wise:

$$\mathbf{u} + \mathbf{v} = (1 + 4,\ -2 + 0,\ 3 + (-1)) = \boxed{(5, -2, 2)},$$

$$\mathbf{u} - \mathbf{v} = (1 - 4,\ -2 - 0,\ 3 - (-1)) = \boxed{(-3, -2, 4)},$$

$$c\mathbf{v} = -2(4, 0, -1) = \boxed{(-8, 0, 2)}.$$

Problem 3.1.1. *Let* $\mathbf{a} = (2, 1, -3)$ *and* $\mathbf{b} = (-1, 4, 0)$*.*

(a) Compute $\mathbf{a} + \mathbf{b}$*.*

(b) Compute $3\mathbf{a} - 2\mathbf{b}$*.*

(c) Find a scalar k *such that* $\mathbf{a} + k\mathbf{b} = (5, 0, -3)$*.*

Solution. *(a)* For vector addition, we add component-wise:

$$\mathbf{a} + \mathbf{b} = (2 + (-1),\ 1 + 4,\ -3 + 0) = \boxed{(1, 5, -3).}$$

(b) First we compute the scaled vectors:

$$3\mathbf{a} = 3(2, 1, -3) = (6, 3, -9), \quad 2\mathbf{b} = 2(-1, 4, 0) = (-2, 8, 0).$$

Then the final answer is

$$\mathbf{3a} - \mathbf{2b} = (6, 3, -9) - (-2, 8, 0) = (6 + 2,\ 3 - 8,\ -9 - 0) = \boxed{(8, -5, -9).}$$

(c) We want

$$\mathbf{a} + k\mathbf{b} = (5, 0, -3).$$

Write this component-wise:

$$(2 + k(-1),\ 1 + 4k,\ -3 + 0 \cdot k) = (5, 0, -3).$$

This gives the system

$$\begin{cases} 2 - k = 5, \\ 1 + 4k = 0, \\ -3 = -3. \end{cases}$$

From the first equation, $2 - k = 5 \Rightarrow k = -3$. From the second, $1 + 4k = 0 \Rightarrow k = -\frac{1}{4}$. These do not agree, so there is no scalar k that satisfies both equations. Therefore, such a scalar does not exist. □

The zero vector and additive inverses.

Definition 3.1.3. *The* zero vector *in* $\mathbb{R}^3$ *is*

$$\mathbf{0} = (0, 0, 0).$$

For a vector $\mathbf{v} = (v_1, v_2, v_3)$*, the* additive inverse *(or negative) of* $\mathbf{v}$ *is*

$$-\mathbf{v} = (-v_1, -v_2, -v_3).$$

These definitions guarantee:

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \quad \mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

Geometrically:

- $\mathbf{0}$ is an arrow of length zero at the origin (or you can think of it as "no displacement").
- $-\mathbf{v}$ is the same arrow as $\mathbf{v}$ but pointing in the opposite direction.

Problem 3.1.2. *Let* $\mathbf{v} = (-3, 5, 1)$*.*

(a) Compute $-\mathbf{v}$*.*

(b) Verify that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$*.*

Solution. *(a)* We simply negate all the components of the original vector $\mathbf{v}$:

$$-\mathbf{v} = -(-3, 5, 1) = \boxed{(3, -5, -1).}$$

(b) We need to add these two vectors and see if we get the zero vector in $\mathbb{R}^3$:

$$\mathbf{v} + (-\mathbf{v}) = (-3, 5, 1) + (3, -5, -1) = (0, 0, 0) = \mathbf{0}.$$

and we have verified the premises of the problem statement. □

Vector space properties in $\mathbb{R}^3$. The set $\mathbb{R}^3$ with the operations of vector addition and scalar multiplication satisfies a list of properties (closure, associativity, distributivity, existence of zero, etc.) that make it a *vector space*. We will later abstract this idea to other spaces. For now, you can think of the following as always true for vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and scalars $a, b \in \mathbb{R}$:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity).
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity).
- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (distributivity).
- $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ (distributivity).
- $1 \cdot \mathbf{v} = \mathbf{v}$.
- There is a zero vector $\mathbf{0}$ with $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- For every $\mathbf{v}$, there is an additive inverse $-\mathbf{v}$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

You can check these using component-wise definitions, but we will not prove all of them here.

The dot product, length, and distance. To measure the length of a vector or the angle between two vectors, we need an operation called the *dot product*.

Definition 3.1.4. *Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$. The* dot product *(or* inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Using the dot product, we define the length (or *magnitude*) of a vector.

Definition 3.1.5. *The* length *(or* norm*) of a vector $\mathbf{v} = (v_1, v_2, v_3)$ is*

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is defined as

$$\|\mathbf{u} - \mathbf{v}\|.$$

Example 3.1.2: Dot product and length. Suppose we have two vectors

$$\mathbf{u} = (1, 2, 2), \quad \mathbf{v} = (2, 0, -1).$$

Compute $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, and the distance between $\mathbf{u}$ and $\mathbf{v}$.

Solution.

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + 2 \cdot 0 + 2 \cdot (-1) = 2 + 0 - 2 = 0.$$

So $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular).

Next, we determine the magnitude of $\mathbf{u}$:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

Finally, we use component-wise subtraction (negated addition) to compute $\mathbf{u} - \mathbf{v}$:

$$\mathbf{u} - \mathbf{v} = (1 - 2,\ 2 - 0,\ 2 - (-1)) = (-1, 2, 3),$$

so the distance is the norm of $\mathbf{u} - \mathbf{v}$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}.$$

Problem 3.1.3. *Let* $\mathbf{a} = (3, -1, 4)$ *and* $\mathbf{b} = (1, 2, -2)$*.*

(a) Compute $\mathbf{a} \cdot \mathbf{b}$*.*

(b) Compute $\|\mathbf{a}\|$*.*

(c) Find the distance between $\mathbf{a}$ *and* $\mathbf{b}$*.*

Solution. *(a)* The dot product is found using component-wise multiplication:

$$\mathbf{a} \cdot \mathbf{b} = 3 \cdot 1 + (-1) \cdot 2 + 4 \cdot (-2) = 3 - 2 - 8 = \boxed{-7}.$$

(b) The length of $\mathbf{a}$ is its magnitude:

$$\|\mathbf{a}\| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{9+1+16} = \boxed{\sqrt{26}}.$$

(c) To find the distance between $\mathbf{a}$ and $\mathbf{b}$, we subtract the vectors and then take the length of the result:

$$\mathbf{a} - \mathbf{b} = (3-1,\ -1-2,\ 4-(-2)) = (2, -3, 6),$$
$$\therefore \|\mathbf{a} - \mathbf{b}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4+9+36} = \sqrt{49} = \boxed{7}.$$

Unit vectors and direction.

Definition 3.1.6. *A vector* $\mathbf{u} \in \mathbb{R}^3$ *is a* unit vector *if* $\|\mathbf{u}\| = 1$*.*

Given a nonzero vector $\mathbf{v}$, we can always form a unit vector in the same direction by dividing by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

This process is also called as *normalizing* a vector.

Problem 3.1.4. *Let* $\mathbf{v} = (2, 2, 1)$*.*

(a) Compute $\|\mathbf{v}\|$*.*

(b) Find a unit vector in the direction of $\mathbf{v}$*.*

Solution. *(a)* The length of the vector is

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = \boxed{3}.$$

(b) A unit vector $\mathbf{u}$ in the same direction is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}(2, 2, 1) = \boxed{\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)}.$$

Angle between two vectors. Finally, the dot product gives a formula for the angle between vectors.

Lemma 3.1.1. *If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are nonzero vectors and θ is the angle between them, then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

Equivalently, the formula reads

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

We will not prove this formula here; you can interpret it as a three-dimensional generalization of the familiar law of cosines from trigonometry.

Problem 3.1.5. *Let $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (0, 1, 1)$. Find the radian measure of the angle between $\mathbf{u}$ and $\mathbf{v}$.*

Solution. First compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

Next, find the lengths of $\mathbf{u}$ and $\mathbf{v}$:

$$\|\mathbf{u}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}, \quad \|\mathbf{v}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}.$$

So Lemma 3.1.1 guarantees

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{1}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2}.$$

So the answer is

$$\theta = \arccos\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}.$$

In this section, we developed a concrete understanding of vectors in $\mathbb{R}^3$ as arrows and ordered triples, and we introduced basic operations like addition, scalar multiplication, length, and the dot product. In the next sections, we will move from $\mathbb{R}^3$ to $\mathbb{R}^n$, and then to more abstract vector spaces that behave like $\mathbb{R}^n$ but consist of objects such as real-valued functions or polynomials.

3.2 The Vector Space $\mathbb{R}^n$ and Subspaces

In Section 3.1, we focused on $\mathbb{R}^3$, specifically, concrete three-dimensional vectors, their operations, and geometric interpretation. In this section, we generalize everything to $\mathbb{R}^n$, where instead of triples we have n-tuples of real numbers. Even though we cannot visualize dimensions higher than 3, the algebra works in exactly the same way.

We will:

- Define $\mathbb{R}^n$ and its vector space structure.
- Introduce the notion of a *subspace* of $\mathbb{R}^n$.
- See important examples of subspaces: lines and planes through the origin, solution sets of homogeneous systems, column spaces and null spaces of matrices.
- Practice checking whether a given set is a subspace.

The vector space $\mathbb{R}^n$.

Definition 3.2.1. *For a positive integer n, the set $\mathbb{R}^n$ is the collection of all ordered n-tuples of real numbers:*

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}.$$

An element of $\mathbb{R}^n$ is called a vector *in $\mathbb{R}^n$.*

We usually write vectors as column vectors:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Definition 3.2.2. *For vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, and a scalar $c \in \mathbb{R}$, we define two operations:*

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n),$$

$$c\mathbf{v} = (cv_1, \ldots, cv_n).$$

These are the natural generalizations of addition and scalar multiplication from three-dimensional space, $\mathbb{R}^3$, to n-dimensional space, $\mathbb{R}^n$.

The zero vector and negatives in $\mathbb{R}^n$.

Definition 3.2.3. *The* zero vector *in $\mathbb{R}^n$ is*

$$\mathbf{0} = (0, 0, \ldots, 0).$$

For $\mathbf{v} = (v_1, \ldots, v_n)$, the negative *(or additive inverse) of $\mathbf{v}$ is*

$$-\mathbf{v} = (-v_1, \ldots, -v_n).$$

Then, as before,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \quad \mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

Example 3.2.1: Basic operations in $\mathbb{R}^4$. Let

$$\mathbf{u} = (1, 0, -2, 3), \quad \mathbf{v} = (-1, 4, 1, 0), \quad c = 2.$$

Compute $\mathbf{u} + \mathbf{v}$, $c\mathbf{u}$, and $\mathbf{u} - \mathbf{v}$.

Solution.

$$\mathbf{u} + \mathbf{v} = (1 + (-1),\ 0 + 4,\ -2 + 1,\ 3 + 0) = \boxed{(0, 4, -1, 3)},$$

$$c\mathbf{u} = 2(1, 0, -2, 3) = \boxed{(2, 0, -4, 6)},$$

$$\mathbf{u} - \mathbf{v} = (1 - (-1),\ 0 - 4,\ -2 - 1,\ 3 - 0) = \boxed{(2, -4, -3, 3)}.$$

Vector space properties (axioms) in $\mathbb{R}^n$. The set $\mathbb{R}^n$ with these operations satisfies the same properties (axioms) as $\mathbb{R}^3$:

- Addition is commutative and associative.
- Scalar multiplication distributes over vector addition and scalar addition.
- There is a zero vector, and each vector has an additive inverse.
- Multiplying by 1 leaves vectors unchanged.

Because of these properties, we say that $\mathbb{R}^n$ is a *vector space*. Later we will see other examples of vector spaces that are not just $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$. Within a vector space, certain subsets behave like smaller vector spaces of their own, as long as they are closed under addition and scalar multiplication.

Definition 3.2.4. *A subset $W \subset \mathbb{R}^n$ is called a* subspace *of $\mathbb{R}^n$ if:*

- *The zero vector $\mathbf{0}$ is in W.*
- *For any $\mathbf{u}, \mathbf{v} \in W$, the sum $\mathbf{u} + \mathbf{v}$ is in W (closed under addition).*
- *For any $\mathbf{v} \in W$ and any scalar $c \in \mathbb{R}$, the vector $c\mathbf{v}$ is in W (closed under scalar multiplication).*

If these conditions hold, then W is itself a vector space with the same operations as $\mathbb{R}^n$.

A subspace is a "flat" subset that contains the origin and is closed under linear combinations. Examples include:

- The whole space $\mathbb{R}^n$.
- The zero subspace $\{\mathbf{0}\}$.
- Lines and planes through the origin in $\mathbb{R}^3$.
- Solution sets of homogeneous linear systems.

Quick subspace test. You often see the following condensed test in practice:

Lemma 3.2.1. *A nonempty subset $W \subset \mathbb{R}^n$ is a subspace if and only if for every $\mathbf{u}, \mathbf{v} \in W$ and every scalar $c \in \mathbb{R}$, the vector*

$$\mathbf{u} + c\mathbf{v}$$

is in W.

Proof. If W is a subspace, it is closed under addition and scalar multiplication. So $\mathbf{u} \in W$ and $c\mathbf{v} \in W$ imply $\mathbf{u} + c\mathbf{v} \in W$.

Conversely, assume W is nonempty and closed under $\mathbf{u} + c\mathbf{v}$. First, let $\mathbf{w} \in W$ be any vector (exists since $W \neq \emptyset$). Take $\mathbf{u} = \mathbf{w}$ and $c = -1$, $\mathbf{v} = \mathbf{w}$. Then

$$\mathbf{u} + c\mathbf{v} = \mathbf{w} + (-1)\mathbf{w} = \mathbf{0} \in W,$$

so W contains the zero vector.

Next, let $\mathbf{u}, \mathbf{v} \in W$ and $c \in \mathbb{R}$. Then:

- Setting $\mathbf{u} = \mathbf{0}$ shows $c\mathbf{v} \in W$ (closed under scalar multiples).
- Setting $c = 1$ shows $\mathbf{u} + \mathbf{v} \in W$ (closed under addition).

Thus W satisfies the subspace conditions. □

Example 3.2.2: A plane through the origin in $\mathbb{R}^3$. Consider the set

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

We claim W is a subspace of $\mathbb{R}^3$. Prove or disprove this statement.

Solution.

- *Zero vector:* For $(0, 0, 0)$, we have $0 + 0 + 0 = 0$, so $\mathbf{0} \in W$.
- *Closed under addition:* Let $\mathbf{u} = (x_1, y_1, z_1) \in W$ and $\mathbf{v} = (x_2, y_2, z_2) \in W$. Then

  $$x_1 + y_1 + z_1 = 0, \quad x_2 + y_2 + z_2 = 0.$$

  Their sum is

  $$\mathbf{u} + \mathbf{v} = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2).$$

  Then we have

  $$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0,$$

  so $\mathbf{u} + \mathbf{v} \in W$.
- *Closed under scalar multiplication:* Let $\mathbf{u} = (x_1, y_1, z_1) \in W$, so $x_1 + y_1 + z_1 = 0$, and let $c \in \mathbb{R}$. Then

  $$c\mathbf{u} = (cx_1, cy_1, cz_1),$$

  and

  $$cx_1 + cy_1 + cz_1 = c(x_1 + y_1 + z_1) = c \cdot 0 = 0,$$

  so $c\mathbf{u} \in W$.

Therefore W is a subspace of $\mathbb{R}^3$. This is a plane through the origin.

Problem 3.2.1. *Determine whether each of the following sets is a subspace of* $\mathbb{R}^3$.

(a) $W_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 0\}$.

(b) $W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 5\}$.

If it is a subspace, briefly justify; if not, explain why.

Solution. *(a)* For W_1, the equation $x - 2y + 3z = 0$ is homogeneous. The zero vector $(0, 0, 0)$ satisfies $0 - 2 \cdot 0 + 3 \cdot 0 = 0$, so $\mathbf{0} \in W_1$. Sums and scalar multiples of solutions of a homogeneous linear equation are again solutions (the equation is linear). Thus W_1 is a subspace of $\mathbb{R}^3$. □

(b) For W_2, the equation is $x - 2y + 3z = 5$, which is not homogeneous. The zero vector gives $0 - 2 \cdot 0 + 3 \cdot 0 = 0 \neq 5$, so $\mathbf{0} \notin W_2$. A subspace must contain the zero vector, so W_2 cannot be a subspace $\mathbb{R}^3$. □

Subspaces as solution sets of homogeneous systems. An important source of subspaces comes from homogeneous systems of linear equations. Consider a system

$$A\mathbf{x} = \mathbf{0},$$

where A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$.

Definition 3.2.5. *The set of all solutions of the homogeneous system* $A\mathbf{x} = \mathbf{0}$ *is called the* null space *of* A*, denoted* $\mathrm{N}(A)$ *or* $\mathrm{null}(A)$.

Lemma 3.2.2. *For any matrix* A*, the null space* $\mathrm{N}(A)$ *is a subspace of* $\mathbb{R}^n$.

Proof. $\mathbf{0}$ is a guaranteed (trivial) solution because $A\mathbf{0} = \mathbf{0}$. If $\mathbf{u}, \mathbf{v}$ are solutions, then

$$A\mathbf{u} = \mathbf{0}, \quad A\mathbf{v} = \mathbf{0}.$$

For any scalar c,

$$A(\mathbf{u} + c\mathbf{v}) = A\mathbf{u} + cA\mathbf{v} = \mathbf{0} + c\mathbf{0} = \mathbf{0},$$

so $\mathbf{u} + c\mathbf{v}$ is also a solution. Thus the solution set is closed under $\mathbf{u} + c\mathbf{v}$, and hence under addition and scalar multiplication. Therefore, the null space is a subspace. □

Example 3.2.3: Null space of a 2×3 matrix. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Find a description of the null space of A, i.e., all $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{0}$.

Solution. The system $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 + 2x_2 + x_3 = 0, \\ x_2 - x_3 = 0. \end{cases}$$

From the second equation, $x_2 = x_3$. Substituting into the first equation yields:

$$x_1 + 2x_2 + x_3 = x_1 + 2x_3 + x_3 = x_1 + 3x_3 = 0 \Rightarrow x_1 = -3x_3.$$

Let $x_3 = t$ be a free parameter. Then

$$x_1 = -3t, \quad x_2 = t, \quad x_3 = t.$$

So the solution vector is given by:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Thus the null space is the set of all scalar multiples of $(-3, 1, 1)^T$:

$$\boxed{\mathrm{N}(A) = \left\{ t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.}$$

Remark. This is a line through the origin in $\mathbb{R}^3$.

Subspaces spanned by vectors. Another important way to get subspaces is to take all linear combinations of some fixed vectors.

Definition 3.2.6. *Given vectors* $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$*, the set of all linear combinations*

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k, \quad c_1, \dots, c_k \in \mathbb{R},$$

is called the span *of* $\mathbf{v}_1, \dots, \mathbf{v}_k$ *and is denoted by*

$$\mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}.$$

Lemma 3.2.3. *For any vectors* $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$*, the set* $\mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is a subspace of the vector space* $\mathbb{R}^n$*.*

Proof. $\mathbf{0}$ is in the span (take all coefficients zero). If $\mathbf{u}$ and $\mathbf{w}$ are linear combinations of the $\mathbf{v}_i$, and c is a scalar, then $\mathbf{u} + c\mathbf{w}$ is also a linear combination of the $\mathbf{v}_i$. Thus the span is closed under $\mathbf{u} + c\mathbf{w}$, and hence is a subspace. □

Example 3.2.4: Span in $\mathbb{R}^3$. Let

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (0, 1, 1).$$

Describe the subspace $W = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution. A general vector in W has the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1(1, 0, 1) + c_2(0, 1, 1) = (c_1,\ c_2,\ c_1 + c_2).$$

Letting $x = c_1$ and $y = c_2$, we have

$$\mathbf{x} = (x, y, x + y),$$

so $z = x + y$. Thus

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid z = x + y\}.$$

This is a plane through the origin in $\mathbb{R}^3$. So we determine that

$$\boxed{W = \{(x, y, z) \mid z = x + y\} \text{ is a subspace of } \mathbb{R}^3.}$$

Problem 3.2.2. *Let*

$$\mathbf{v}_1 = (1, 2, 0), \quad \mathbf{v}_2 = (0, 1, 3) \in \mathbb{R}^3.$$

(a) Show that $W = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ *is a subspace of* $\mathbb{R}^3$*.*

(b) Describe W *as the solution set of a single linear equation* $ax + by + cz = 0$*.*

Solution. *(a)* According to Lemma 3.2.3, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is always a subspace of $\mathbb{R}^3$. No additional work is needed here.

(b) A general vector in W is

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1(1, 2, 0) + c_2(0, 1, 3) = (c_1,\ 2c_1 + c_2,\ 3c_2).$$

Write $\mathbf{x} = (x, y, z)$. Then

$$x = c_1, \quad z = 3c_2 \Rightarrow c_2 = \frac{z}{3}, \quad y = 2c_1 + c_2 = 2x + \frac{z}{3}.$$

So any $(x, y, z) \in W$ must satisfy

$$y = 2x + \frac{z}{3} \quad \Rightarrow \quad 3y = 6x + z \quad \Rightarrow \quad 6x - 3y + z = 0.$$

Thus we obtain

$$\boxed{W = \{(x, y, z) \mid 6x - 3y + z = 0\}.}$$

In this section, we generalized vectors from $\mathbb{R}^3$ to $\mathbb{R}^n$ and introduced the crucial concept of a *subspace*. We saw that:

- Subspaces are subsets of $\mathbb{R}^n$ closed under addition and scalar multiplication.
- Solution sets of homogeneous systems $A\mathbf{x} = \mathbf{0}$ are subspaces called null spaces.
- Spans of vectors are subspaces and give lines and planes (and higher-dimensional analogues) through the origin.

In the next section, we will dig deeper into linear combinations, span, and linear independence of vectors.

3.3 Linear Combinations and Independence of Vectors

In the previous section, we saw that many important subspaces of $\mathbb{R}^n$ can be described as spans of vectors or as solution sets of homogeneous systems. In this section, we make the ideas of *span*, *linear combination*, and *linear independence* precise.

We will:

- Define linear combinations and the span of a set of vectors.
- Learn how to decide whether a vector is in the span of others.
- Define linear independence and dependence.
- Use systems of equations / row reduction to test independence.

Linear combinations and span.

Definition 3.3.1. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^n$*. A vector* $\mathbf{w} \in \mathbb{R}^n$ *is called a* linear combination *of* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *if there exist scalars* $c_1, \ldots, c_k \in \mathbb{R}$ *such that*

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k.$$

Definition 3.3.2. *The set of all linear combinations of* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *is called their* span*, denoted*

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}.$$

As we saw in Section 3.2, the span is always a subspace of $\mathbb{R}^n$. You can think of the span as "all directions and magnitudes you can reach" by combining the given vectors (taking possible linear combinations).

Example 3.3.1: Linear combinations in $\mathbb{R}^3$. Suppose we have the two vectors

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (0, 1, 1).$$

Is $\mathbf{w} = (2, 3, 5)$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$?

Solution. We ask whether there exist scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}.$$

That is, we wish to demonstrate

$$c_1(1, 0, 1) + c_2(0, 1, 1) = (2, 3, 5),$$

so simplifying, we have

$$(c_1,\ c_2,\ c_1 + c_2) = (2, 3, 5).$$

This gives the linear system

$$\begin{cases} c_1 = 2, \\ c_2 = 3, \\ c_1 + c_2 = 5. \end{cases}$$

The first two equations give $c_1 + c_2 = 2 + 3 = 5$, which matches the third equation. Thus there is a solution: $c_1 = 2,\ c_2 = 3$.

So the vector $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2.$$

and we have demonstrated that $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. □

Problem 3.3.1. *Let*

$$\mathbf{v}_1 = (1, 2, 0), \quad \mathbf{v}_2 = (0, 1, 1), \quad \mathbf{w} = (1, 3, 2).$$

Determine whether $\mathbf{w}$ *is a linear combination of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*. If so, find scalars* c_1, c_2 *such that* $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

Solution. We want to find $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1,2,0) + c_2(0,1,1) = (1,3,2).$$

We simplify the equation:

$$(c_1,\ 2c_1 + c_2,\ c_2) = (1,3,2),$$

which yields the linear system

$$\begin{cases} c_1 = 1, \\ 2c_1 + c_2 = 3, \\ c_2 = 2. \end{cases}$$

From $c_1 = 1$ and $c_2 = 2$, the middle equation becomes $2(1)+2 = 4 \neq 3$, so the system is inconsistent. Therefore, there are no scalars c_1, c_2 with $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, so

$$\boxed{\mathbf{w} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}.}$$

Checking membership in a span using matrices. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w} \in \mathbb{R}^n$, deciding whether $\mathbf{w}$ is in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ amounts to solving

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{w}.$$

We can write this as a linear system:

$$\begin{pmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{w}.$$

So we place the $\mathbf{v}_i$ as columns of a matrix A and solve $A\mathbf{c} = \mathbf{w}$ for $\mathbf{c}$.

Example 3.3.2: Using row-reduction to test span. Suppose we have the vectors

$$\mathbf{v}_1 = (1,1,0), \quad \mathbf{v}_2 = (2,0,1), \quad \mathbf{v}_3 = (1,-1,1), \quad \mathbf{w} = (3,2,1).$$

Is $\mathbf{w}$ in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Solution. We form the augmented matrix for

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}.$$

That is,

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

We need to apply row operations to reduce the matrix. We start with the original matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Applying $R_2 \leftarrow R_2 - R_1$ gives:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -2 & -2 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Next, we apply $R_3 \leftarrow R_3 + \frac{1}{2}R_2$ to obtain:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

At this point, the last row is represented by the equation

$$0c_1 + 0c_2 + 0c_3 = \frac{1}{2},$$

which is impossible, because $0 = \frac{1}{2}$ is a contradiction. So the system has no solution. Therefore, $\mathbf{w}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and cannot span the set of these three vectors:

$$\boxed{\mathbf{w} \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.}$$

Linear independence and dependence. The span of a vector set tells us what such vectors can generate. Linear independence tells us whether any of those vectors are redundant.

Definition 3.3.3. *The vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ *are* linearly independent *if the only solution to the following*

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

is $c_1 = \cdots = c_k = 0$ *(the trivial solution).*

If there exists a nontrivial solution (some $c_i \neq 0$*) to this equation, then the vectors are called* linearly dependent*.*

Intuitively:

- Independent vectors do not "point in the same direction combination-wise". None can be written as a linear combination of the others.
- Dependent vectors have redundancy: at least one of them can be written as a combination of the others.

Example 3.3.3: Dependent vectors in $\mathbb{R}^3$. Let

$$\mathbf{v}_1 = (1, 2, 3), \quad \mathbf{v}_2 = (2, 4, 6), \quad \mathbf{v}_3 = (1, 0, 1).$$

Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent or dependent?

Solution. Notice that $\mathbf{v}_2 = 2\mathbf{v}_1$. Then

$$2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3 = 2\mathbf{v}_1 - 2\mathbf{v}_1 = \mathbf{0},$$

with coefficients $(2, -1, 0)$ not all zero. Therefore, the set is linearly dependent.

Problem 3.3.2. *Determine whether each set of vectors is linearly independent or dependent. If dependent, find a nontrivial linear relation.*

(a) $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$ *in* $\mathbb{R}^3$.

(b) $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (2, 2)$ *in* $\mathbb{R}^2$.

Solution. *(a)* Suppose we wish to solve the equation

$$c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0).$$

This gives $c_1=0,\ c_2=0,\ c_3=0$. So the only solution is the trivial one, and the vectors are linearly independent. □

(b) Suppose we wish to solve the equation

$$c_1(1,1)+c_2(2,2)=(0,0).$$

Then

$$(c_1+2c_2,\ c_1+2c_2)=(0,0)\Rightarrow c_1+2c_2=0.$$

This has infinitely many solutions, for example $c_1=2,\ c_2=-1$. So

$$2(1,1)+(-1)(2,2)=(0,0),$$

and the vectors are linearly dependent. □

Row-reduction test for independence. To test whether $\mathbf{v}_1,\ldots,\mathbf{v}_k$ are linearly independent in $\mathbb{R}^n$, we set up the homogeneous system

$$c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k=\mathbf{0}.$$

Writing the $\mathbf{v}_i$ as columns of a matrix A, this is

$$A\mathbf{c}=\mathbf{0},\quad \mathbf{c}=(c_1,\ldots,c_k)^T.$$

The vectors are independent if and only if the only solution is $\mathbf{c}=\mathbf{0}$, which happens exactly when the columns of A are pivot columns (no free variables in the homogeneous system).

Example 3.3.4: Testing independence with row reduction. Let

$$\mathbf{v}_1=(1,2,1),\quad \mathbf{v}_2=(0,1,1),\quad \mathbf{v}_3=(2,5,3).$$

Are these vectors linearly independent?

Solution. Form the matrix with these as columns:

$$A=\begin{pmatrix}1&0&2\\2&1&5\\1&1&3\end{pmatrix}.$$

Row-reduce A to see whether the columns are independent. We start with the row operation $R_2\leftarrow R_2-2R_1$:

$$\begin{pmatrix}1&0&2\\0&1&1\\1&1&3\end{pmatrix}$$

We then apply $R_3 \leftarrow R_3 - R_1$:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

To handle the last row, we perform $R_3 \leftarrow R_3 - R_2$:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that the third row is all zero, so there are only two pivots (in columns 1 and 2). Column 3 is a linear combination of the first two columns, which means $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus the set is linearly dependent. □

We can even find a relation by solving the matrix equation

$$A\mathbf{c} = \mathbf{0}.$$

From the reduced form,

$$\begin{cases} c_1 + 2c_3 = 0, \\ c_2 + c_3 = 0. \end{cases}$$

Let $c_3 = t$. Then $c_1 = -2t$, $c_2 = -t$. Taking $t = 1$, we obtain

$$(-2)\mathbf{v}_1 + (-1)\mathbf{v}_2 + (1)\mathbf{v}_3 = \mathbf{0},$$

or an alternate form

$$\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2.$$

So we know that $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Problem 3.3.3. *Determine whether the following vectors in* $\mathbb{R}^3$ *are linearly independent:*

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (0, 1, 1), \quad \mathbf{v}_3 = (1, 1, 2).$$

Hint: Put them as columns of a matrix and row-reduce.

Solution. Let A be the matrix obtained by taking the vectors as columns:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

We will apply row reduction. We start with $R_3 \leftarrow R_3 - R_1$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Now we apply $R_3 \leftarrow R_3 - R_2$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Here, we have only two pivots, so the columns are linearly dependent. Solving $A\mathbf{c} = \mathbf{0}$ from the reduced form gives us the following system:

$$\begin{cases} c_1 + c_3 = 0, \\ c_2 + c_3 = 0. \end{cases}$$

Let $c_3 = t$. Then $c_1 = -t$, $c_2 = -t$. Taking $t = 1$, we get

$$(-1)\mathbf{v}_1 + (-1)\mathbf{v}_2 + (1)\mathbf{v}_3 = \mathbf{0},$$

or in alternate form:

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2.$$

The vector $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, thus the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. □

Maximum number of independent vectors in $\mathbb{R}^n$. A fundamental fact (whose full proof we postpone for now) is:

Lemma 3.3.1. *Any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent.*

Intuitively, you cannot have more than n "directions" that are all independent in an n-dimensional space. This idea leads directly to the concepts of *basis* and *dimension*, which we will study in the next section.

Problem 3.3.4. *Decide whether each statement is true or false. Justify briefly.*

(a) Any three vectors in $\mathbb{R}^2$ are linearly dependent.

(b) Any two vectors in $\mathbb{R}^3$ are linearly independent.

Solution. *(a)* This statement is $\boxed{\text{true}}$. By the lemma, any set of more than 2 vectors in $\mathbb{R}^2$ is linearly dependent.

(b) This statement is $\boxed{\text{false}}$. For example, $\mathbf{v}_1 = (1, 0, 0)$ and $\mathbf{v}_2 = (2, 0, 0)$ lie on the same line and satisfy $2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, so they are linearly dependent.

In this section, we formalized the notions of linear combinations, span, and linear independence. These concepts help us:

- Describe subspaces as spans of vectors.
- Detect redundancy in a set of vectors.
- Understand how many vectors are needed to "generate" a subspace.

In the next section, we will use these ideas to define bases and dimensions for vector spaces, which provide a precise measure of "how many directions" a space has.

3.4 Bases and Dimensions for Vector Spaces

In the last section, we learned about linear combinations, span, and linear independence. We saw that some sets of vectors are redundant, while others give just enough "directions" to generate a subspace. In this section, we formalize this idea with the concepts of *basis* and *dimension.*

We will:

- Define a basis of a vector space and see examples in $\mathbb{R}^n$.
- Introduce the notion of dimension as the number of vectors in a basis.
- Learn how to find a basis for the span of given vectors using row reduction.
- Practice computing bases and dimensions of common subspaces (null spaces, column spaces, planes, lines).

What is a basis? Roughly speaking, a basis is a "minimal spanning set" for a vector space: it spans the space, and none of its vectors are redundant.

Definition 3.4.1. *Let V be a vector space. A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in V is called a* basis *of V if two conditions are met:*

- $\mathbf{v}_1, \dots, \mathbf{v}_k$ *span V (every vector in V is a linear combination of them), and*
- $\mathbf{v}_1, \dots, \mathbf{v}_k$ *are linearly independent.*

If you know a basis of a vector space, you have some very valuable information:

- How to generate every vector (using linear combinations).
- That you are not using any more vectors than necessary.

Example 3.4.1: The standard basis for $\mathbb{R}^n$. In $\mathbb{R}^3$, the vectors

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

have a special role: every vector (x, y, z) can be written uniquely as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

These vectors are linearly independent and span $\mathbb{R}^3$, so they form a basis of $\mathbb{R}^3$.

More generally:

Definition 3.4.2. *In $\mathbb{R}^n$, the* standard basis *is the set of vectors*

$$\mathbf{e}_1 = (1, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \ \dots, \quad \mathbf{e}_n = (0, \dots, 0, 1),$$

where $\mathbf{e}_i$ has a 1 in the i-th coordinate and 0 elsewhere.

Lemma 3.4.1. *The standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.*

Proof. Any vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ can be written as

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n,$$

so the standard basis spans $\mathbb{R}^n$. It is also clear that the only way

$$c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n = \mathbf{0}$$

can hold is if each $c_i = 0$ (for $i = 1, 2, 3, \dots, n$), so the vectors are linearly independent. Therefore, the standard basis is indeed a basis. □

Coordinates relative to a basis. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of a vector space V and $\mathbf{x} \in V$, then there are unique scalars $c_1, \ldots, c_k$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k.$$

The scalars $(c_1, \ldots, c_k)$ are called the *coordinates* of $\mathbf{x}$ relative to this basis.

Problem 3.4.1. *Suppose we have two vectors*

$$\mathbf{v}_1 = (1, 1), \quad \mathbf{v}_2 = (1, -1)$$

in the vector space $\mathbb{R}^2$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$, and find the coordinates of $\mathbf{x} = (3, 1)$ relative to this basis.

Solution. First, we check that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. Suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \Rightarrow c_1(1, 1) + c_2(1, -1) = (0, 0).$$

Then simplifying yields

$$(c_1 + c_2,\ c_1 - c_2) = (0, 0) \Rightarrow \begin{cases} c_1 + c_2 = 0, \\ c_1 - c_2 = 0. \end{cases}$$

Adding the two equations gives $2c_1 = 0 \Rightarrow c_1 = 0$, and then $c_2 = 0$. So the set is linearly independent.

Since there are two linearly independent vectors in $\mathbb{R}^2$, they must span $\mathbb{R}^2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis.

Next, we find scalars a, b such that

$$(3, 1) = a(1, 1) + b(1, -1) = (a + b,\ a - b).$$

This gives the linear system

$$\begin{cases} a + b = 3, \\ a - b = 1. \end{cases}$$

Adding, $2a = 4 \Rightarrow a = 2$. Then $b = 3 - a = 1$.

So relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$,

$$(3, 1) = 2\mathbf{v}_1 + 1\mathbf{v}_2.$$

□

Dimension. A key fact (which is not obvious but can be proved with some work) is that any two bases of a given vector space have the same number of vectors. This allows us to define the dimension of a vector space.

Definition 3.4.3. *If a vector space V has a basis consisting of k vectors, we say V is k-*dimensional *and write*

$$\dim V = k.$$

If $V = \{\mathbf{0}\}$ (only the zero vector), we define $\dim V = 0$.

For example:

- $\mathbb{R}^n$ has dimension n.
- Any line through the origin in $\mathbb{R}^3$ has dimension 1.
- Any plane through the origin in $\mathbb{R}^3$ has dimension 2.

Example 3.4.2: Dimension of some subspaces in $\mathbb{R}^3$. Consider the following two subspaces of the vector space $\mathbb{R}^3$:

$$L = \{t(1,2,3) \mid t \in \mathbb{R}\}, \quad P = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}.$$

Solution.

- L is the set of all scalar multiples of the ordered triple $(1,2,3)$. The set $\{(1,2,3)\}$ is linearly independent and spans L, so it forms a basis of L. Since the basis contains one vector, $(1,2,3)$, we have $\boxed{\dim L = 1}$.

- For P, we can find a basis by solving the equation $x+y+z=0$. Let y and z be free parameters:

  $$x = -y - z.$$

  Then any vector in P has the form

  $$(x,y,z) = (-y-z,\ y,\ z) = y(-1,1,0) + z(-1,0,1).$$

  Thus

  $$P = \operatorname{span}\{(-1,1,0),\ (-1,0,1)\}.$$

  These two vectors are linearly independent (you can check this), so they form a basis of P. Hence $\boxed{\dim P = 2}$.

Finding a basis from a spanning set. Suppose you are given some vectors that span a subspace $W \subset \mathbb{R}^n$, but they may be redundant. To find a basis, you want a *subset* of these vectors that:

- still spans the same subspace W, and
- is linearly independent.

We can do this systematically using row reduction.

Method (column picture). Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$, and let A be the matrix whose columns are these vectors. Row-reduce A to its row-echelon form. The columns of A corresponding to pivot columns in the echelon form form a basis of $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Example 3.4.3: Basis from three vectors in $\mathbb{R}^3$. Let

$$\mathbf{v}_1 = (1,2,1), \quad \mathbf{v}_2 = (2,4,2), \quad \mathbf{v}_3 = (1,0,1).$$

Find a basis for $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and compute $\dim W$.

Solution. Form the matrix A with these as columns:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

To solve the system, we row reduce A. start with the operation $R_2 \leftarrow R_2 - 2R_1$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

Then to handle the third row, we carry out $R_3 \leftarrow R_3 - R_1$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Now pivot in column 1 (row 1) and column 3 (row 2). So columns 1 and 3 of the original matrix are pivot columns. Therefore, the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ form a basis of W, and the dimension is $\boxed{\dim W = 2}$.

Bases for null spaces and column spaces. Two especially important subspaces associated with a matrix A are:

- The *null space*, $\mathrm{N}(A)$: all solutions of $A\mathbf{x} = \mathbf{0}$.
- The *column space*, $\mathrm{Col}(A)$: the span of the columns of A.

Example 3.4.4: Basis of a null space. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

We already solved $A\mathbf{x} = \mathbf{0}$ in an earlier example, but we repeat it here briefly.

Solution. Solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 0, \\ x_2 - x_3 = 0. \end{cases}$$

From the second equation, $x_2 = x_3$. Then $x_1 + 3x_3 = 0 \Rightarrow x_1 = -3x_3$. Let $x_3 = t$. Then

$$\mathbf{x} = \begin{pmatrix} -3t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}.$$

Thus

$$\mathrm{N}(A) = \left\{ t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

A basis for the null space is

$$\boxed{\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \right\}},$$

and its dimension is

$$\boxed{\dim \mathrm{N}(A) = 1}$$

Example 3.4.5: Basis of a column space. Consider the same matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Its columns are

$$A_1 = (1,0)^T, \quad A_2 = (2,1)^T, \quad A_3 = (1,-1)^T.$$

We want a basis for $\mathrm{Col}(A) = \mathrm{span}\{A_1, A_2, A_3\}$.

We already row-reduced A earlier:

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

is essentially already in echelon form with pivot columns 1 and 2. Thus A_1 and A_2 form a basis for the column space:

$$\boxed{\text{Basis of Col}(A) : \{(1,0)^T,\ (2,1)^T\}, \quad \dim \mathrm{Col}(A) = 2}$$

In this section, we introduced:

- **Basis**: a linearly independent spanning set.
- **Dimension**: the number of vectors in any basis of a vector space.
- Methods for finding bases for spans, null spaces, and column spaces using row reduction.

In the next section, we will see that many other mathematical objects (like polynomials and matrices) form vector spaces with their own bases and dimensions, extending these ideas beyond $\mathbb{R}^n$.

3.5 Generalized Vector Spaces

So far, our vector spaces have mostly been $\mathbb{R}^n$, where vectors are n-tuples of real numbers. However, many other objects behave like vectors: polynomials, matrices, and even functions can form vector spaces. In this section, we look at these *generalized vector spaces* and see that the same ideas of linear combination, basis, and dimension still apply.

We will:

- Define vector spaces abstractly.
- Give examples: polynomials, matrices, and functions.
- Compute bases and dimensions in these new settings.
- Practice checking whether a given subset is a subspace.

Abstract definition of a vector space. We now step back and describe the properties that make something a "vector space" without referring directly to coordinates.

Definition 3.5.1. *A* vector space *over $\mathbb{R}$ is a set V together with two operations:*

- *vector addition: $V \times V \to V$, $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}$,*
- *scalar multiplication: $\mathbb{R} \times V \to V$, $(c, \mathbf{v}) \mapsto c\mathbf{v}$,*

such that the following properties hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $a, b \in \mathbb{R}$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *(commutativity).*
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *(associativity).*
3. *There is a zero vector $\mathbf{0} \in V$ with $\mathbf{v} + \mathbf{0} = \mathbf{v}$.*
4. *For every $\mathbf{v} \in V$ there is an additive inverse $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.*
5. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
6. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
7. $(ab)\mathbf{v} = a(b\mathbf{v})$.
8. $1 \cdot \mathbf{v} = \mathbf{v}$.

You have already seen that $\mathbb{R}^n$ with standard addition and scalar multiplication satisfies all of these. The point of this abstraction is that many other objects satisfy the same list.

Polynomials as vectors. Let $\mathcal{P}_n$ denote the set of all real polynomials in x of degree at most n:

$$\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, \ldots, a_n \in \mathbb{R}\}.$$

Lemma 3.5.1. *With the usual addition and scalar multiplication of polynomials, $\mathcal{P}_n$ is a vector space.*

Proof. If $p(x)$ and $q(x)$ are polynomials of degree at most n, then $p(x) + q(x)$ is still a polynomial of degree at most n, and $c\,p(x)$ is also of degree at most n for any scalar c. The zero polynomial 0 is in $\mathcal{P}_n$, and every polynomial has an additive inverse $-p(x)$. The remaining vector space properties (commutativity, associativity, distributivity) follow from the usual algebra of polynomials. □

Example 3.5.1: Basis and dimension of $\mathcal{P}_2$. Consider $\mathcal{P}_2$, the set of polynomials of degree at most 2:

$$\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}.$$

A natural set of polynomials is

$$1, \quad x, \quad x^2.$$

Any $p(x) \in \mathcal{P}_2$ can be written uniquely as

$$p(x) = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2,$$

so $\{1, x, x^2\}$ spans $\mathcal{P}_2$, and the coefficients a_0, a_1, a_2 are unique. It is straightforward to check that $c_0 \cdot 1 + c_1 x + c_2 x^2 = 0$ for all x implies $c_0 = c_1 = c_2 = 0$, so the set is linearly independent.

Thus $\{1, x, x^2\}$ is a basis of $\mathcal{P}_2$, and has dimension of 3.

Problem 3.5.1. *Show that $\{1, x, x^2, x^3\}$ is a basis of $\mathcal{P}_3$, and state the dimension of $\mathcal{P}_3$.*

Solution: Any polynomial in $\mathcal{P}_3$ has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

so it is a linear combination of $1, x, x^2, x^3$. The representation is unique: if

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 \equiv 0$$

for all x, then all coefficients must be zero: $c_0 = c_1 = c_2 = c_3 = 0$. Thus $\boxed{\{1, x, x^2, x^3\}}$ is linearly independent and spans $\mathcal{P}_3$, so it is a basis, with dimension $\boxed{\dim \mathcal{P}_3 = 4}$.

Matrices as vectors. Let $M_{m\times n}$ denote the set of all real $m \times n$ matrices:

$$M_{m\times n} = \{A = [a_{ij}] \mid a_{ij} \in \mathbb{R},\ 1 \le i \le m,\ 1 \le j \le n\}.$$

Lemma 3.5.2. *With matrix addition and scalar multiplication defined entry-wise, $M_{m\times n}$ is a vector space.*

Proof. If A and B are $m \times n$ matrices, then $A + B$ is also $m \times n$, and cA is $m \times n$ for any scalar c. The zero matrix (all entries zero) acts as the zero vector, and $-A$ is the additive inverse of A. The remaining axioms follow from the properties of addition and scalar multiplication of real numbers, applied entry-wise. □

Dimension of $M_{2\times 2}$. Consider $M_{2\times 2}$, the set of all 2×2 real matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Each matrix is determined by four real numbers a, b, c, d. We can think of a basis made from matrices that have a single 1 in one position and 0 elsewhere:

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any matrix A can be written as a linear combination of the "standard basis" matrices, i.e. E_{11}, E_{12}, E_{21}, and E_{22}:

$$A = aE_{11} + bE_{12} + cE_{21} + dE_{22}.$$

This representation is unique, and the four E_{ij} are clearly linearly independent. Thus $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis of $M_{2\times 2}$ with dimension $\dim M_{2\times 2} = 4$.

Problem 3.5.2. *Find the dimension of $M_{3\times 2}$, the vector space of all 3×2 real matrices. Describe a natural basis for this space.*

Solution. A 3×2 matrix has $3 \cdot 2 = 6$ total entries. For each position (i, j), we can define E_{ij} to be the matrix with 1 in that position and 0 elsewhere. There are 6 such matrices, and they form a basis of $M_{3\times 2}$, by the same reasoning as with the 2×2 case. Thus the basis is $\boxed{\{E_{11}, E_{12}, E_{21}, E_{22}, E_{31}, E_{32}\}}$ with dimension of $\boxed{\dim M_{3\times 2} = 6}$.

Functions as vectors. Another important class of vector spaces consists of functions. For example, let $C[a, b]$ be the set of all continuous real-valued functions defined on the interval $[a, b]$.

Lemma 3.5.3. *With pointwise addition and scalar multiplication,*

$$(f + g)(x) = f(x) + g(x), \quad (cf)(x) = c\, f(x),$$

the set $C[a, b]$ is a vector space.

Proof. The sum of two continuous functions is continuous, and a scalar multiple of a continuous function is continuous, so $C[a, b]$ is closed under both operations. The zero function $0(x) = 0$ is in $C[a, b]$, and every function has an additive inverse $-f$. The remaining axioms follow from the properties of real numbers, applied pointwise. □

In this course, we will mostly use function spaces conceptually, without computing bases and dimensions (these can be infinite-dimensional).

Subspaces inside generalized vector spaces. Just as in $\mathbb{R}^n$, subsets of these generalized vector spaces can be subspaces if they are closed under addition and scalar multiplication.

Example: Subspace of polynomials. Let

$$W = \{p(x) \in \mathcal{P}_3 \mid p(0) = 0\}.$$

We claim W is a subspace of $\mathcal{P}_3$.

Solution.

- The zero polynomial 0 satisfies $0(0) = 0$, so $0 \in W$.
- If $p(0) = 0$ and $q(0) = 0$, then $(p + q)(0) = p(0) + q(0) = 0$, so $p + q \in W$.
- If $p(0) = 0$ and $c \in \mathbb{R}$, then $(cp)(0) = c\, p(0) = c \cdot 0 = 0$, so $cp \in W$.

Thus W is a subspace.

We can describe W more concretely. If $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, the condition $p(0) = 0$ implies that $a_0 = 0$. So

$$W = \{a_1 x + a_2 x^2 + a_3 x^3 \mid a_1, a_2, a_3 \in \mathbb{R}\}.$$

Naturally, we see the basis $\{x, x^2, x^3\}$, with dimension of $\dim W = 3$.

Problem 3.5.3. *Let $\mathcal{P}_2$ be the vector space of polynomials of maximal degree 2. Determine whether each set is a subspace of $\mathcal{P}_2$:*

(a) $W_1 = \{p(x) \in \mathcal{P}_2 \mid p(1) = 0\}$.

(b) $W_2 = \{p(x) \in \mathcal{P}_2 \mid p(1) = 1\}$.

If it is a subspace, find a basis and the dimension.

Solution. *(a)* For W_1, we check the subspace conditions:

- The zero polynomial 0 satisfies $0(1) = 0$, so $0 \in W_1$.
- If $p(1) = 0$ and $q(1) = 0$, then $(p+q)(1) = p(1) + q(1) = 0$, so $p + q \in W_1$.
- If $p(1) = 0$ and $c \in \mathbb{R}$, then $(cp)(1) = c\,p(1) = c \cdot 0 = 0$, so $cp \in W_1$.

So we know W_1 is a subspace. Now to describe W_1. A general $p(x) \in \mathcal{P}_2$ has the form

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

The condition $p(1) = 0$ gives

$$a_0 + a_1 + a_2 = 0 \Rightarrow a_0 = -a_1 - a_2.$$

So we write the polynomial as

$$p(x) = -a_1 - a_2 + a_1 x + a_2 x^2 = a_1(x-1) + a_2(x^2 - 1).$$

So we have

$$W_1 = \operatorname{span}\{x - 1,\ x^2 - 1\}.$$

The polynomials $x - 1$ and $x^2 - 1$ are not multiples of each other, so they are linearly independent. So $\boxed{\{x-1,\ x^2-1\}}$ is a basis of W_1, and W_1 is a subspace of $\mathcal{P}_2$, with dimension $\boxed{\dim W_1 = 2}$.

(b) For W_2, the zero polynomial satisfies $0(1) = 0 \neq 1$, so $0 \notin W_2$. Since a subspace must contain the zero vector, W_2 cannot be a subspace. □

Remark. The zero vector represents the zero polynomial, $p(x) \equiv 0$ in this case.

Problem 3.5.4. *Let S be the set of all symmetric 2×2 matrices:*

$$S = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}.$$

Show that S is a subspace of $M_{2\times 2}$ and find its dimension.

Solution.

- The zero matrix is symmetric, so $\mathbf{0} \in S$.
- The sum of symmetric matrices is symmetric, and a scalar multiple of a symmetric matrix is symmetric, so S is closed under addition and scalar multiplication.

Thus S is a subspace of $M_{2\times 2}$. Also, any $A \in S$ has the form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So S is spanned by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These three matrices are linearly independent, so they form a basis of S. Therefore, the dimension of S is $\boxed{\dim S = 3}$.

4 Higher-Order Differential Equations

A higher-order differential equation is a differential equation involving derivatives of order two or higher. While first-order equations describe systems governed by a single rate of change, higher–order equations arise naturally in models involving acceleration, curvature, oscillation, and wave propagation.

4.1 Second-Order Homogeneous Linear Equations

In the previous chapter we studied first-order differential equations and developed several techniques for solving them. Many important physical models, however, lead naturally to differential equations involving a *second derivative.* For example, Newton's second law states that the acceleration of a particle is proportional to the force acting on it, which leads to equations of the form

$$my'' = F(x, y, y').$$

Because of their importance in physics, engineering, and applied mathematics, differential equations involving second derivatives form one of the central topics in the theory of differential equations.

In this section we introduce *second-order linear differential equations* and focus on the homogeneous case. Understanding the structure of their solutions will allow us to develop systematic methods for solving many important equations later in the chapter.

Second-Order Linear Differential Equations

Definition 4.1.1. *A* second-order differential equation *for an unknown function $y = y(x)$ is an equation involving y, its first derivative y', and its second derivative y''.*

It is called linear *if it can be written in the form*

$$y'' + p(x)y' + q(x)y = f(x),$$

where $p(x)$, $q(x)$, and $f(x)$ are given functions of the independent variable x.

When $f(x) \equiv 0$, we obtain the homogeneous form

$$y'' + p(x)y' + q(x)y = 0.$$

Definition 4.1.2. *A second-order linear differential equation is called* homogeneous *if*

$$y'' + p(x)y' + q(x)y = 0.$$

If $f(x) \not\equiv 0$, the equation is called nonhomogeneous.

Problem 4.1.1. *Determine whether the following equations are linear and whether they are homogeneous.*

1. $y'' + 3y' + 2y = 0$
2. $y'' + y^2 = 0$
3. $y'' - xy' + 4y = e^x$

Solution.

1. $y'' + 3y' + 2y = 0$ is linear and homogeneous.
2. $y'' + y^2 = 0$ is not linear because y^2 appears.
3. $y'' - xy' + 4y = e^x$ is linear but nonhomogeneous.

Initial Value Problems For first-order equations, specifying one initial condition determines a unique solution. For second-order equations, two conditions are required.

Definition 4.1.3. *An* initial value problem *for*

$$y'' + p(x)y' + q(x)y = f(x)$$

consists of the equation together with two conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'.$$

These conditions determine the initial position and slope of the solution curve.

Problem 4.1.2. *Explain why two initial conditions are required to determine a unique solution of a second-order differential equation.*

Solution. A second-order differential equation involves the second derivative y'', so two integrations are required to obtain the solution. Each integration introduces a constant of integration. Therefore two constants must be determined, requiring two initial conditions.

Existence and Uniqueness

Theorem 4.1.1. *Let $p(x)$ and $q(x)$ be continuous on an interval I. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = y_0, \qquad y'(x_0) = y_0'$$

has a unique solution on I.

This theorem guarantees that once two initial conditions are specified, exactly one solution exists.

The Trivial Solution

Definition 4.1.4. *The function*

$$y(x) \equiv 0$$

is called the trivial solution *of a homogeneous differential equation.*

Problem 4.1.3. *Verify that $y(x) = 0$ satisfies*

$$y'' + p(x)y' + q(x)y = 0$$

for any functions $p(x)$ and $q(x)$.

Solution. If $y(x) = 0$, then

$$y'(x) = 0, \qquad y''(x) = 0.$$

Substituting into the equation gives

$$0 + p(x) \cdot 0 + q(x) \cdot 0 = 0.$$

Thus $y(x) = 0$ satisfies the equation.

The Superposition Principle

Lemma 4.1.1. *Let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0.$$

Then the function

$$y = c_1 y_1 + c_2 y_2$$

is also a solution.

Problem 4.1.4. *Show that if y_1 and y_2 satisfy the homogeneous equation, then $2y_1 - 3y_2$ is also a solution.*

Solution. By the superposition principle, any linear combination

$$y = c_1 y_1 + c_2 y_2$$

is a solution. Choosing $c_1 = 2$ and $c_2 = -3$ gives

$$y = 2y_1 - 3y_2$$

which is therefore also a solution.

Linear Independence

Definition 4.1.5. *Functions y_1 and y_2 are* linearly independent *on an interval I if*

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for all $x \in I$ implies $c_1 = c_2 = 0$.

Problem 4.1.5. *Determine whether the following pairs of functions are linearly independent.*

1. e^x, e^{2x}
2. e^x, $3e^x$

Solution.

1. e^x and e^{2x} are **linearly independent** because no constant multiple of one equals the other.
2. e^x and $3e^x$ are **linearly dependent** since $3e^x = 3 \cdot e^x$.

The Wronskian

Definition 4.1.6. *The* Wronskian *of a set of two functions* $\{y_1, y_2\}$ *is*

$$W\{y_1, y_2\}(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Problem 4.1.6. *Compute the Wronskian of* $y_1 = e^x$ *and* $y_2 = e^{-x}$.

Solution. The Wronskian is given by the determinant

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = \boxed{-2}.$$

Since $W \neq 0$, the functions are linearly independent.

Structuring the General Solution

Theorem 4.1.2. *If* y_1 *and* y_2 *are linearly independent solutions of*

$$y'' + p(x)y' + q(x)y = 0,$$

then every solution can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Problem 4.1.7. *Suppose* $y_1 = e^x$ *and* $y_2 = e^{-x}$ *are solutions of* $y'' - y = 0$. *Write the general solution.*

Solution. The general solution is a linear combination of the two linearly independent solutions.

$$\boxed{y(x) = c_1 e^x + c_2 e^{-x}}.$$

Problem 4.1.8. *Solve the initial value problem*

$$y'' - y = 0, \quad y(0) = 3, \quad y'(0) = 1.$$

Solution. We know from the previous problem that the general solution to the equation $y''-y=0$ is

$$y = c_1e^x + c_2e^{-x}.$$

Also, we find y':

$$y' = c_1e^x - c_2e^{-x}.$$

Applying initial conditions, we obtain the following linear system:

$$\begin{cases} c_1 + c_2 = 3 \\ c_1 - c_2 = 1 \end{cases}$$

Solving the system yields

$$c_1 = 2, \qquad c_2 = 1.$$

Thus the particular solution is

$$\boxed{y(x) = 2e^x + e^{-x}}.$$

4.2 Introduction to Higher-Order Linear Equations

In the previous section we studied second-order homogeneous linear differential equations and discovered an important structural property: the set of solutions behaves like a vector space. In particular, any linear combination of solutions is again a solution.

Many important models lead to differential equations of order higher than two. For example, beam deflection in structural engineering involves fourth-order equations, while certain control systems and electrical circuits can lead to equations of order three or higher. Fortunately, many of the ideas we developed for second-order equations extend naturally to higher-order equations.

Higher-Order Differential Equations

Definition 4.2.1. *An* n-th order differential equation *is an equation involving an unknown function* $y(x)$ *and its derivatives up to order* n*:*

$$y,\ y',\ y'',\ \dots,\ y^{(n)}.$$

Definition 4.2.2. *An* n*-th order differential equation is called* linear *if it can be written in the form*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

where the functions $a_0(x), \dots, a_{n-1}(x)$ *and* $f(x)$ *depend only on the independent variable* x*.*

If the right-hand side is zero, we obtain the homogeneous equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

Definition 4.2.3. *A linear differential equation is called* homogeneous *if*

$$f(x) = 0.$$

Otherwise it is called nonhomogeneous.

Problem 4.2.1. *Determine whether the following differential equations are linear. If they are linear, determine whether they are homogeneous.*

1. $y''' + 2y'' - y = 0$
2. $y''' + y^2 = 0$
3. $y'''' + xy'' + y = e^x$

Solution. To determine whether an equation is linear, we check whether the unknown function and its derivatives appear only to the first power.

1. $y''' + 2y'' - y = 0$

   All terms involve derivatives of y linearly, and the right-hand side is zero. Thus the equation is **linear and homogeneous**.

2. $y''' + y^2 = 0$

   The term y^2 makes the equation nonlinear, so the equation is **not linear**.

3. $y'''' + xy'' + y = e^x$

   The derivatives appear linearly, so the equation is **linear**. However the right-hand side is not zero, so the equation is **nonhomogeneous**.

Initial Value Problems For higher-order equations, more initial conditions are required.

Definition 4.2.4. *An* initial value problem *for an n-th order differential equation consists of the equation together with n conditions:*

$$y(x_0) = c_0, \quad y'(x_0) = c_1, \quad \ldots, \quad y^{(n-1)}(x_0) = c_{n-1}.$$

These conditions determine the value of the function and its first $n - 1$ derivatives at the initial point.

Problem 4.2.2. *How many initial conditions are required to determine a unique solution of a fourth-order differential equation?*

Solution. A fourth-order equation involves the derivative $y^{(4)}$. Solving the equation requires four integrations to obtain y, introducing four constants of integration. Therefore four initial conditions are required to determine these constants uniquely.

Superposition Principle The most important property of homogeneous linear equations extends naturally to higher-order equations.

Theorem 4.2.1 (Superposition Principle)**.** *If $y_1, y_2, \ldots, y_k$ are solutions of the homogeneous equation*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0,$$

then any linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ky_k$$

is also a solution.

Problem 4.2.3. *Suppose y_1 and y_2 are solutions of a third-order homogeneous equation. Show that $5y_1 - 2y_2$ is also a solution.*

Solution. The superposition principle tells us that any linear combination of solutions must also satisfy the differential equation.

Since $5y_1 - 2y_2$ is simply the linear combination with coefficients $c_1 = 5$ and $c_2 = -2$, it follows that this function must also be a solution. □

Linear Independence of Solutions Just as in the second-order case, the concept of linear independence plays an important role.

Definition 4.2.5. *Functions $y_1, y_2, \ldots, y_n$ are* linearly independent *on an interval I if*

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

for all $x \in I$ implies that

$$c_1 = c_2 = \cdots = c_n = 0.$$

Problem 4.2.4. *Determine whether the following functions are linearly independent on the interval $I = (-\infty, \infty)$:*

$$e^x, \quad e^{2x}, \quad e^{3x}.$$

Solution. To check independence we ask whether one function can be written as a constant multiple of the others.

Observe that none of these functions is a constant multiple of another:

$$\frac{e^{2x}}{e^x} = e^x, \qquad \frac{e^{3x}}{e^x} = e^{2x}.$$

Since these ratios depend on x, the functions cannot be constant multiples of one another. Thus the functions are linearly independent on I. □

The Wronskian for Higher-Order Equations To test independence of several solutions we use the Wronskian.

Definition 4.2.6. *The* Wronskian *of a set of functions $y_1, \ldots, y_n$ is defined as the determinant*

$$W\{y_1, \ldots, y_n\}(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

If this determinant is nonzero at some point of the interval, the functions are linearly independent.

Problem 4.2.5. *Compute the Wronskian of*

$$y_1 = e^x, \quad y_2 = e^{2x}.$$

Solution. First compute derivatives:

$$y_1' = e^x, \qquad y_2' = 2e^{2x}.$$

The Wronskian is

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}.$$

which simplifies to.

$$W = 2e^{3x} - e^{3x} = \boxed{e^{3x}}.$$

Since $e^{3x} \neq 0$, the functions are linearly independent.

General Solution Structure The following result generalizes what we observed for second-order equations.

Theorem 4.2.2. *Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the homogeneous equation*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

Then every solution can be written as

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x).$$

Thus the space of solutions has a dimension of n.

Problem 4.2.6. *Suppose $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are solutions of a third-order equation. Write the general solution.*

Solution. Because the three functions are linearly independent, every solution must be a linear combination of them. Therefore the general solution is

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}}.$$

The constants $c_1, c_2, c_3 \in \mathbb{R}$ are determined by the initial conditions.

In this section we introduced higher-order linear differential equations and examined the structure of their solution spaces. The key idea is that homogeneous linear equations allow superposition, and their solution sets behave like vector spaces of dimension equal to the order of the equation.

In the next section we begin developing practical techniques for solving such equations, starting with the important case of linear equations with constant coefficients.

4.3 Second-Order Constant Coefficient Homogeneous Equations

In the previous section we introduced higher-order linear differential equations and discussed the structure of their solution spaces. We now develop a practical technique for solving one of the most important classes of equations: second-order homogeneous linear equations with constant coefficients.

Equations of this form arise frequently in applications, including models of oscillations, mechanical vibrations, electrical circuits, and population dynamics.

Definition 4.3.1. *A* second-order constant coefficient homogeneous differential equation *is an equation of the form*

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants and $a \neq 0$.

Our goal is to determine all functions $y(x)$ that satisfy this equation.

The Exponential Trial Solution Because the coefficients of the equation are constant, it is natural to look for solutions whose derivatives behave in a predictable way. The exponential function has this property.

Suppose we attempt a solution of the form

$$y = e^{rx},$$

where r is a constant to be determined. Then we have the first- and second-order derivatives

$$y' = re^{rx}, \qquad y'' = r^2 e^{rx}.$$

Substituting into the differential equation gives

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Factoring out e^{rx} yields

$$e^{rx}(ar^2 + br + c) = 0.$$

Since e^{rx} is never zero, we must have

$$ar^2 + br + c = 0.$$

Definition 4.3.2. *The equation*

$$ar^2 + br + c = 0$$

is called the characteristic equation *or* characteristic polynomial *of the differential equation*

$$ay'' + by' + cy = 0.$$

Thus solving the differential equation reduces to solving an algebraic quadratic equation, i.e. finding the roots of the characteristic polynomial.

Problem 4.3.1. *Find the characteristic equation of*

$$y'' - 7y' + 10y = 0.$$

Solution. We compare the equation with the standard form

$$ay'' + by' + cy = 0.$$

Here $a = 1$, $b = -7$, and $c = 10$. Therefore the characteristic polynomial is

$$\boxed{r^2 - 7r + 10 = 0}.$$

Three Possible Cases The roots of the quadratic equation determine the form of the solution. Since the equation is quadratic, there are three possibilities:

- two distinct real roots
- one repeated real root
- two complex conjugate roots

We examine each case separately.

Case 1: Distinct Real Roots Suppose the characteristic equation has two distinct real roots r_1 and r_2.

Then the functions

$$y_1 = e^{r_1 x}, \qquad y_2 = e^{r_2 x}$$

are solutions of the differential equation.

Because these functions are linearly independent, the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 4.3.1. *Solve*

$$y'' - 5y' + 6y = 0.$$

Solution. The characteristic equation is

$$r^2 - 5r + 6 = 0.$$

Factoring, we have

$$(r - 2)(r - 3) = 0.$$

The roots of the polynomial are

$$r_1 = 2, \qquad r_2 = 3.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{3x}}.$$

Problem 4.3.2. *Solve the differential equation*

$$y'' - 4y' + 3y = 0.$$

Solution. The characteristic equation is

$$r^2 - 4r + 3 = 0.$$

Factoring gives

$$(r - 1)(r - 3) = 0.$$

We observe that the roots are $r_1 = 1$ and $r_2 = 3$. Therefore, the general solution is

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{3x}}.$$

Case 2: Repeated Roots Suppose the characteristic equation has a repeated root r.
In this case the function e^{rx} is a solution, but we need a second linearly independent solution. It turns out that the function

$$y_2 = xe^{rx}$$

provides the required second solution, as attaching a linear term x is sufficient to ensure that both solutions are linearly independent.

Theorem 4.3.1. *If the characteristic equation has a repeated root r, then the general solution of the differential equation*

$$ay'' + by' + cy = 0$$

is the function

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}.$$

Example 4.3.2. *Find the general solution to the differential equation*

$$y'' - 4y' + 4y = 0.$$

Solution. The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

Factoring, we have

$$(r-2)^2 = 0.$$

Thus we have the root $r = 2$ with multiplicity 2.

So the general solution becomes

$$c_1 e^{2x} + c_2 x e^{2x} \quad \Longrightarrow \quad \boxed{y(x) = (c_1 + c_2 x)e^{2x}}.$$

Problem 4.3.3. *Solve the differential equation*

$$y'' + 6y' + 9y = 0.$$

Solution. The characteristic equation is

$$r^2 + 6r + 9 = 0.$$

Factoring,

$$(r+3)^2 = 0.$$

Thus $r = -3$ is a repeated root (has multiplicity 2). The general solution becomes

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} \quad \Longrightarrow \quad \boxed{y(x) = (c_1 + c_2 x)e^{-3x}}.$$

Case 3: Complex Roots Suppose the characteristic equation has complex roots

$$r = \alpha \pm i\beta.$$

Using Euler's formula, the corresponding real solutions are written as

$$e^{\alpha x}\cos(\beta x), \qquad e^{\alpha x}\sin(\beta x).$$

Theorem 4.3.2. *If the characteristic equation has complex roots $\alpha \pm i\beta$, then the general solution is*

$$y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right).$$

Example 4.3.3. *Solve the differential equation*

$$y'' + y = 0.$$

Solution. The characteristic equation is

$$r^2 + 1 = 0.$$

There are two complex roots:

$$r = \pm i.$$

Here, we have $\alpha = 0$ and $\beta = 1$, so the general solution is

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x}.$$

Problem 4.3.4. *Find the general solution to the differential equation*

$$y'' + 4y = 0.$$

Solution. The characteristic equation is

$$r^2 + 4 = 0.$$

This polynomial has two complex roots

$$r = \pm 2i.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 \cos(2x) + c_2 \sin(2x)}.$$

Solving Initial Value Problems Once the general solution is known, we determine the constants using the initial conditions.

Example 4.3.4. *Solve*

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. The general solution is

$$y = c_1 \cos x + c_2 \sin x.$$

Compute the derivative:

$$y' = -c_1 \sin x + c_2 \cos x.$$

Apply the initial conditions.

$$y(0) = c_1 = 1$$

$$y'(0) = c_2 = 0$$

Thus the general solution is

$$\boxed{y(x) = \cos x}.$$

Practice Problems Here, we will simply carry out the mathematical process, as we have communicated the ideas behind solving previous similar exercises.

Problem 4.3.5. *Solve the following differential equations.*

1. $y'' - 3y' - 10y = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 9y = 0$
4. $y'' - 2y' + 5y = 0$

Solution.

1. $r^2 - 3r - 10 = 0$

$$(r-5)(r+2) = 0$$

$$\therefore \boxed{y = c_1e^{5x} + c_2e^{-2x}}$$

2. $r^2 + 2r + 1 = 0$

$$(r+1)^2 = 0$$

$$\therefore \boxed{y = (c_1 + c_2x)e^{-x}}$$

3. $r^2 + 9 = 0$

$$r = \pm 3i$$

$$\therefore \boxed{y = c_1 \cos 3x + c_2 \sin 3x}$$

4. $r^2 - 2r + 5 = 0$

$$r = 1 \pm 2i$$

$$\therefore \boxed{y = e^x(c_1 \cos 2x + c_2 \sin 2x)}$$

In this section we developed a method for solving second-order homogeneous equations with constant coefficients. By introducing an exponential trial solution, we reduced the differential equation to a quadratic algebraic equation. The nature of the roots of the characteristic equation determines the form of the solution.

In the next section we will extend these ideas to nonhomogeneous equations, where additional forcing terms appear on the right-hand side.

4.4 Higher-Order Constant Coefficient Homogeneous Equations

In the previous section we developed a method for solving second-order homogeneous differential equations with constant coefficients by introducing an exponential trial solution. The key idea was to reduce the differential equation to an algebraic equation whose roots determine the form of the solution.

In this section we extend that method to differential equations of arbitrary order. Remarkably, the same exponential approach works for higher-order equations as well.

General Form

Definition 4.4.1. *An n-th order* constant coefficient homogeneous differential equation *has the form*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

Our goal is to determine all functions $y(x)$ that satisfy this equation.

Exponential Trial Solutions As before, we attempt a solution of the form

$$y = e^{rx}.$$

Then

$$y' = re^{rx}, \quad y'' = r^2 e^{rx}, \quad \ldots \quad y^{(n)} = r^n e^{rx}.$$

Substituting into the differential equation gives

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0.$$

Factoring out e^{rx} yields

$$e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) = 0.$$

Since $e^{rx} \neq 0$, we obtain the polynomial equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

Definition 4.4.2. *The polynomial*

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0$$

is called the characteristic equation *of the differential equation.*

Thus solving the differential equation reduces to solving a polynomial equation.

Problem 4.4.1. *Find the characteristic equation of*

$$y''' - 6y'' + 11y' - 6y = 0.$$

Solution. Replacing $y^{(k)}$ by r^k gives the polynomial

$$\boxed{r^3 - 6r^2 + 11r - 6 = 0}.$$

This is the characteristic equation.

Structure of the Solutions Suppose the characteristic equation has n roots

$$r_1, r_2, \dots, r_n.$$

Each root produces a solution of the differential equation.
If all roots are distinct, the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}.$$

Problem 4.4.2. *Find the general solution to the differential equation*

$$y''' - 6y'' + 11y' - 6y = 0.$$

Solution. The characteristic equation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Factoring gives

$$(r-1)(r-2)(r-3) = 0.$$

The roots are

$$r_1 = 1, \quad r_2 = 2, \quad r_3 = 3.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}.$$

Repeated Roots If a root has multiplicity greater than one, additional independent solutions appear.

Suppose r is a root of multiplicity k. Then the following k functions are the linearly independent solutions:

$$e^{rx}, \quad xe^{rx}, \quad x^2e^{rx}, \quad \dots, \quad x^{k-1}e^{rx}.$$

Theorem 4.4.1. *If the characteristic equation has a root r of multiplicity k, then the differential equation has k linearly independent solutions*

$$e^{rx}, \ xe^{rx}, \ x^2e^{rx}, \ \dots, \ x^{k-1}e^{rx}.$$

Problem 4.4.3. *Find the general solution to the differential equation*

$$y''' - 3y'' + 3y' - y = 0.$$

Solution. The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0.$$

Recognizing the binomial expansion,

$$(r-1)^3 = 0.$$

Thus $r = 1$ is a root of multiplicity 3.

The general solution is therefore the linear combination of all independent solutions:

$$\boxed{y(x) = c_1e^x + c_2xe^x + c_3x^2e^x}.$$

Complex Roots If the characteristic equation has complex roots of the form

$$r = \alpha \pm i\beta,$$

then the corresponding real solutions are

$$e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x).$$

Problem 4.4.4. *Find the general solution to the differential equation*

$$y''' + y' = 0.$$

Solution. The characteristic equation is

$$r^3 + r = 0.$$

Factoring gives

$$r(r^2 + 1) = 0.$$

The complex (one real, two pure imaginary) roots of the polynomial are

$$r_1 = 0, \quad r_2 = i, \quad r_3 = -i.$$

Euler's formula gives the corresponding real-valued solutions:

$$1, \quad \cos x, \quad \sin x.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 + c_2 \cos x + c_3 \sin x}.$$

Combining Multiple Types of Roots In many equations the characteristic polynomial involves several root classifications.

Problem 4.4.5. *Solve*

$$y^{(4)} - 2y''' + y'' = 0.$$

Solution. The characteristic equation is

$$r^4 - 2r^3 + r^2 = 0.$$

Factoring gives:

$$r^2(r-1)^2 = 0.$$

We have the following roots:

$$r = 0 \text{ (multiplicity 2)}, \qquad r = 1 \text{ (multiplicity 2)}.$$

The corresponding solutions are

$$1, \quad x, \quad e^x, \quad xe^x.$$

So the general solution is

$$\boxed{y(x) = c_1 + c_2 x + c_3 e^x + c_4 x e^x}.$$

Solving Initial Value Problems Once the general solution is known, the constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$ are determined by the given initial conditions.

Problem 4.4.6. *Find the general solution to the differential equation*

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = 0.$$

Solution. The general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

To determine the constants, we compute the first- and second-order derivatives:

$$y' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$$

$$y'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$$

Applying the initial conditions, we have the system

$$\begin{cases} c_1 + c_2 + c_3 = 2 \\ c_1 + 2c_2 + 3c_3 = 1 \\ c_1 + 4c_2 + 9c_3 = 0 \end{cases}$$

Solving the system using Gaussian elimination yields

$$c_1 = 3, \quad c_2 = -3, \quad c_3 = 2.$$

Thus, the particular solution is

$$\boxed{y(x) = 3e^x - 3e^{2x} + 2e^{3x}}.$$

Practice Problems Here, we will simply carry out the mathematical process, as we have communicated the ideas behind solving previous similar exercises.

Problem 4.4.7. *Solve the following differential equations.*

1. $y''' - 7y'' + 16y' - 12y = 0$
2. $y''' + 3y'' + 3y' + y = 0$
3. $y^{(4)} + 2y'' + y = 0$
4. $y^{(4)} - y'' = 0$

Solution.

1. $r^3 - 7r^2 + 16r - 12 = 0$

$$(r-2)^2(r-3) = 0$$

$$\therefore \boxed{y = (c_1 + c_2 x)e^{2x} + c_3 e^{3x}}$$

2. $r^3 + 3r^2 + 3r + 1 = 0$

$$(r+1)^3 = 0$$

$$\therefore \boxed{y = (c_1 + c_2 x + c_3 x^2)e^{-x}}$$

3. $r^4 + 2r^2 + 1 = 0$

$$(r^2+1)^2 = 0$$

$$\therefore \boxed{y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x}$$

4. $r^4 - r^2 = 0$

$$r^2(r^2 - 1) = 0$$

$$r = 0, 0, 1, -1$$

$$\therefore \boxed{y = c_1 + c_2 x + c_3 e^{x} + c_4 e^{-x}}$$

In this section we extended the characteristic equation method to higher-order constant coefficient differential equations. The key idea is that the roots of the characteristic polynomial determine the structure of the solution. Repeated roots produce polynomial factors of x, while complex roots lead to oscillatory solutions involving sine and cosine functions.

In the next section we turn to nonhomogeneous linear differential equations, where an additional forcing term appears on the right-hand side.

4.5 Method of Undetermined Coefficients

In the previous sections we studied homogeneous linear differential equations and learned how to solve constant coefficient equations by means of the characteristic equation. We now turn to the nonhomogeneous case, where an additional forcing term appears on the right-hand side.

Our goal in this section is to solve equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x),$$

with particular emphasis on second-order equations such as

$$ay'' + by' + cy = g(x).$$

A central principle from linear differential equations is that the general solution of a nonhomogeneous equation has the form

$$y = y_h + y_p,$$

where:

- y_h is the general solution of the associated homogeneous equation, and
- y_p is any one particular solution of the nonhomogeneous equation.

So the entire problem reduces to finding a suitable particular solution y_p.

Why Guessing Can Work At first, it may seem unreasonable to try to *guess* a particular solution. But for certain forcing functions $g(x)$, the derivatives of $g(x)$ remain in the same general family of functions. For example:

- the derivative of a polynomial is another polynomial,
- the derivative of e^{ax} is a constant multiple of e^{ax},
- the derivatives of $\sin bx$ and $\cos bx$ cycle among themselves,
- the derivatives of $e^{ax} \cos bx$ and $e^{ax} \sin bx$ remain linear combinations of those same functions.

This observation suggests a practical strategy: if the forcing term belongs to one of these families, we can try a particular solution of the same general shape, with unknown coefficients. These coefficients are then determined by substitution into the differential equation. This is the *method of undetermined coefficients*.

The Basic Idea Consider a nonhomogeneous linear differential equation

$$L[y] = g(x),$$

where L is a linear differential operator with constant coefficients. The method of undetermined coefficients proceeds in three stages:

1. Solve the associated homogeneous equation $L[y] = 0$ to find y_h.
2. Guess a trial form for y_p based on the shape and behavior of $g(x)$.

3. Substitute the trial form into the differential equation and solve for the unknown coefficients.

A crucial warning is needed here: the trial form for y_p must be chosen so that it does *not* overlap with the homogeneous solution. If it does, we must multiply by a sufficiently large power of x to make it linearly independent from the homogeneous part. This is the phenomenon often called *resonance*.

Case 1: Polynomial Forcing Terms Suppose the forcing term is a polynomial, for example

$$g(x) = 3x^2 - 5x + 1.$$

Since derivatives of polynomials are still polynomials, it is natural to try a polynomial particular solution of the same degree:

$$y_p = Ax^2 + Bx + C.$$

Example 4.5.1. *Solve the differential equation*

$$y'' - 3y' + 2y = x^2.$$

Solution. As always, we begin with the homogeneous equation:

$$y'' - 3y' + 2y = 0.$$

Its characteristic equation is

$$r^2 - 3r + 2 = 0,$$

which factors as

$$(r-1)(r-2) = 0.$$

So the homogeneous solution is

$$y_h = c_1e^x + c_2e^{2x}.$$

Now the forcing term is the polynomial x^2. What kind of trial function should we try? Since differentiating polynomials keeps us inside the family of polynomials, it is reasonable to try

$$y_p = Ax^2 + Bx + C.$$

Compute the first- and second-order derivatives:

$$y_p' = 2Ax + B, \qquad y_p'' = 2A.$$

Substitute into the differential equation:

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2.$$

Now expand and collect like terms:

$$2Ax^2 + (-6A + 2B)x + (2A - 3B + 2C) = x^2.$$

For two polynomials to be equal for all x, their coefficients must match. So we compare coefficients:

$$2A = 1, \qquad -6A + 2B = 0, \qquad 2A - 3B + 2C = 0.$$

From the first equation,

$$A = \frac{1}{2}.$$

Then

$$-6\left(\frac{1}{2}\right) + 2B = 0 \implies -3 + 2B = 0 \implies B = \frac{3}{2}.$$

Finally,

$$2\left(\frac{1}{2}\right) - 3\left(\frac{3}{2}\right) + 2C = 0$$

gives

$$1 - \frac{9}{2} + 2C = 0 \implies 2C = \frac{7}{2} \implies C = \frac{7}{4}.$$

Therefore

$$y_p = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}.$$

So the general solution is

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x} + \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}.}$$

Problem 4.5.1. *Solve the differential equation*

$$y'' + y = 3x - 1.$$

Solution. First we solve the homogeneous equation:

$$y'' + y = 0.$$

The characteristic equation is

$$r^2 + 1 = 0,$$

so the roots are

$$r = \pm i.$$

Hence the solution to the homogeneous solution is

$$y_h = c_1 \cos x + c_2 \sin x.$$

Now the forcing term is the polynomial $3x - 1$, so we try

$$y_p = Ax + B.$$

We find the first- and second-order derivatives:

$$y_p' = A, \qquad y_p'' = 0.$$

Substitute into the equation:

$$0 + (Ax + B) = 3x - 1.$$

This is pleasantly simple. Matching coefficients gives

$$A = 3, \qquad B = -1.$$

So the particular solution is

$$y_p = 3x - 1,$$

and the overall general solution is

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x + 3x - 1.}$$

Case 2: Exponential Forcing Terms Suppose

$$g(x) = e^{ax}.$$

Since derivatives of e^{ax} are constant multiples of e^{ax}, the natural trial form is

$$y_p = Ae^{ax}.$$

Example 4.5.2. *Solve the differential equation*

$$y'' - y = e^{2x}.$$

Solution. The homogeneous equation is

$$y'' - y = 0,$$

with characteristic equation

$$r^2 - 1 = 0.$$

So

$$r = \pm 1,$$

and

$$y_h = c_1 e^x + c_2 e^{-x}.$$

Now the forcing term is e^{2x}, and this does not overlap with the homogeneous solution, so try

$$y_p = Ae^{2x}.$$

Then

$$y_p' = 2Ae^{2x}, \qquad y_p'' = 4Ae^{2x}.$$

Substitute:

$$4Ae^{2x} - Ae^{2x} = e^{2x}.$$

So

$$3Ae^{2x} = e^{2x}.$$

Since $e^{2x} \neq 0$,

$$3A = 1, \qquad A = \frac{1}{3}.$$

So the particular solution is

$$y_p = \frac{1}{3}e^{2x},$$

and therefore the full solution is

$$\boxed{y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3}e^{2x}.}$$

Problem 4.5.2. *Solve the differential equation*

$$y'' + 4y' + 4y = e^{-x}.$$

Solution. The homogeneous equation has characteristic equation

$$r^2 + 4r + 4 = 0,$$

which factors as

$$(r + 2)^2 = 0.$$

Thus the homogeneous solution is

$$y_h = (c_1 + c_2 x)e^{-2x}.$$

The forcing term is e^{-x}. Since e^{-x} is not part of the homogeneous solution, we try guessing

$$y_p = Ae^{-x}.$$

The first and second derivatives of y_p are

$$y_p' = -Ae^{-x}, \qquad y_p'' = Ae^{-x}.$$

Substituting into the original differential equation, we have

$$Ae^{-x} + 4(-Ae^{-x}) + 4Ae^{-x} = e^{-x}.$$

The left-hand side simplifies to

$$Ae^{-x}.$$

So we have

$$Ae^{-x} = e^{-x},$$

hence $A = 1$.

Therefore the overall solution is

$$\boxed{y(x) = (c_1 + c_2 x)e^{-2x} + e^{-x}.}$$

Case 3: Trigonometric Forcing Terms If

$$g(x) = \cos bx \quad \text{or} \quad g(x) = \sin bx,$$

then derivatives cycle between sine and cosine. Therefore we must include both in the trial function:

$$y_p = A \cos bx + B \sin bx.$$

Example 4.5.3. *Solve*

$$y'' + y = \cos x.$$

Solution. First solve the homogeneous equation:

$$y'' + y = 0.$$

Its characteristic roots are $\pm i$, so

$$y_h = c_1 \cos x + c_2 \sin x.$$

Now here is the subtle point: if we try

$$y_p = A \cos x + B \sin x,$$

then this trial function duplicates the homogeneous solution. That means it can never produce a genuinely new particular solution. So we must multiply by x to force linear independence.

Thus we try

$$y_p = x(A \cos x + B \sin x).$$

Let us expand this first:

$$y_p = Ax \cos x + Bx \sin x.$$

Differentiate:

$$y_p' = A \cos x - Ax \sin x + B \sin x + Bx \cos x,$$

and

$$y_p'' = -2A \sin x - Ax \cos x + 2B \cos x - Bx \sin x.$$

Now substitute into $y'' + y$:

$$y_p'' + y_p = \big(-2A \sin x - Ax \cos x + 2B \cos x - Bx \sin x\big) + \big(Ax \cos x + Bx \sin x\big).$$

The $x \cos x$ and $x \sin x$ terms cancel, leaving

$$y_p'' + y_p = -2A \sin x + 2B \cos x.$$

We want this to equal $\cos x$, so compare coefficients:

$$-2A = 0, \qquad 2B = 1.$$

Thus

$$A = 0, \qquad B = \frac{1}{2}.$$

So

$$y_p = \frac{1}{2} x \sin x.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2} x \sin x.}$$

Problem 4.5.3. *Solve*

$$y'' + 4y = \sin 2x.$$

Solution. The homogeneous equation is

$$y'' + 4y = 0,$$

whose characteristic equation is

$$r^2 + 4 = 0.$$

Thus

$$r = \pm 2i,$$

and

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

If we tried

$$y_p = A \cos 2x + B \sin 2x,$$

we would again duplicate the homogeneous solution. So resonance is present, and we multiply by x:

$$y_p = x(A \cos 2x + B \sin 2x).$$

At this point, instead of differentiating blindly, it is worth asking: what are we hoping for? We want the substituted expression to simplify to something involving only $\sin 2x$ and $\cos 2x$, so the trial form is appropriate.

Carrying out the differentiation and substitution gives

$$y_p'' + 4y_p = -4A \sin 2x + 4B \cos 2x.$$

We want this to equal $\sin 2x$, so

$$-4A = 1, \qquad 4B = 0.$$

Hence

$$A = -\frac{1}{4}, \qquad B = 0.$$

So the particular solution is

$$y_p = -\frac{1}{4} x \cos 2x,$$

and therefore

$$\boxed{y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} x \cos 2x.}$$

Case 4: Exponential Times Polynomial If the forcing term has the form

$$g(x) = e^{ax} P_n(x),$$

where $P_n(x)$ is a polynomial of degree n, then the trial form should be

$$y_p = e^{ax}(A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0).$$

Example 4.5.4. *Solve*

$$y'' - 2y' + y = xe^x.$$

Solution. First solve the homogeneous equation:

$$y'' - 2y' + y = 0.$$

The characteristic equation is

$$r^2 - 2r + 1 = 0,$$

so factoring gives

$$(r-1)^2 = 0.$$

Thus the homogeneous solution is

$$y_h = (c_1 + c_2 x)e^x.$$

Now the forcing term is xe^x. The naive trial form would be

$$y_p = e^x(Ax + B).$$

But notice something important: both e^x and xe^x already appear in the homogeneous solution. That means our trial form overlaps completely with y_h. So we must multiply by a large enough power of x to eliminate the overlap.

Since the root $r = 1$ has multiplicity 2, we multiply by x^2:

$$y_p = x^2 e^x(Ax + B).$$

That is,

$$y_p = e^x(Ax^3 + Bx^2).$$

At this stage the computation becomes longer, but the idea remains the same: differentiate, substitute, and match coefficients. Carrying out the algebra gives

$$A = \frac{1}{6}, \qquad B = 0.$$

Thus the particular solution is

$$y_p = \frac{1}{6}x^3 e^x.$$

So the general solution is

$$\boxed{y(x) = (c_1 + c_2 x)e^x + \frac{1}{6}x^3 e^x.}$$

Case 5: Exponential Times Trigonometric Functions If

$$g(x) = e^{ax}\cos bx \quad \text{or} \quad g(x) = e^{ax}\sin bx,$$

then the correct trial form is

$$y_p = e^{ax}(A\cos bx + B\sin bx).$$

Example 4.5.5. *Solve the differential equation*

$$y'' - 2y' + 2y = e^x \cos x.$$

Solution. The characteristic equation is

$$r^2 - 2r + 2 = 0.$$

Its roots are

$$r = 1 \pm i.$$

Therefore

$$y_h = e^x(c_1 \cos x + c_2 \sin x).$$

Now look carefully at the forcing term:

$$e^x \cos x.$$

The natural trial form

$$e^x(A \cos x + B \sin x)$$

is already part of the homogeneous solution. So resonance occurs, and we must multiply by the least term, x, to guarantee linear independence:

$$y_p = xe^x(A \cos x + B \sin x).$$

Substituting this trial function into the equation and simplifying yields

$$A = 0, \qquad B = \frac{1}{2}.$$

Thus the particular solution is

$$y_p = \frac{1}{2}xe^x \sin x,$$

and so the general solution is

$$\boxed{y(x) = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}xe^x \sin x.}$$

Summary of Trial Forms The method of undetermined coefficients applies when the forcing term $g(x)$ is built from combinations of the following types:

Forcing term $g(x)$	Trial form for y_p
Polynomial $P_n(x)$	$A_n x^n + \cdots + A_0$
e^{ax}	Ae^{ax}
$\cos bx$ or $\sin bx$	$A \cos bx + B \sin bx$
$e^{ax}P_n(x)$	$e^{ax}(A_n x^n + \cdots + A_0)$
$e^{ax} \cos bx$ or $e^{ax} \sin bx$	$e^{ax}(A \cos bx + B \sin bx)$

If any part of the trial form overlaps with the homogeneous solution, multiply the entire trial form by x^s, where s is large enough to remove the overlap.

The Resonance Rule Here is the rule students most often forget:

> If the naive trial solution duplicates part of the homogeneous solution, multiply the trial form by enough powers of x to make it linearly independent from y_h.

This is not an optional adjustment. Without it, the method fails immediately.

Problem 4.5.4. *For each forcing term below, write down an appropriate trial form for y_p. Do not solve for coefficients.*

1. $y'' - 3y' + 2y = x^2$
2. $y'' - 2y' + y = e^x$
3. $y'' + 9y = \cos 3x$
4. $y'' - 4y' + 13y = e^{2x}\sin 3x$
5. $y'' - 2y' + y = xe^x$

Solution.

1. For x^2, we should try a polynomial with degree at most 2:
$$y_p = Ax^2 + Bx + C.$$
2. Since the homogeneous equation has repeated root $r = 1$, the homogeneous solution contains e^x and xe^x. The naive trial Ae^x overlaps, so multiply by x^2:
$$y_p = Ax^2e^x.$$
3. Since the homogeneous equation has roots $\pm 3i$, the homogeneous solution already contains $\cos 3x$ and $\sin 3x$. So multiply by x:
$$y_p = x(A\cos 3x + B\sin 3x).$$
4. The natural trial form is
$$y_p = e^{2x}(A\cos 3x + B\sin 3x).$$
Since the homogeneous roots are $2 \pm 3i$, resonance occurs, so multiply by x:
$$y_p = xe^{2x}(A\cos 3x + B\sin 3x).$$
5. The forcing term is xe^x. Since $r = 1$ is a repeated root of the homogeneous equation, we multiply the naive form by x^2:
$$y_p = x^2e^x(Ax + B).$$

Linearity and Sums of Forcing Terms If the forcing term is a sum

$$g(x) = g_1(x) + g_2(x) + \cdots + g_m(x),$$

we may look for a particular solution of the form

$$y_p = y_{p,1} + y_{p,2} + \cdots + y_{p,m},$$

where each $y_{p,j}$ is chosen to match $g_j(x)$.

This works because the differential operator is linear.

Example 4.5.6. *Solve*

$$y'' + y = x + \sin x.$$

Solution. First solve the homogeneous equation:

$$y'' + y = 0,$$

so the general solution

$$y_h = c_1 \cos x + c_2 \sin x.$$

Now the forcing term is a sum:

$$x + \sin x.$$

We split the particular solution as

$$y_p = y_{p,1} + y_{p,2},$$

where:

$$y_{p,1} \text{ corresponds to } x, \qquad y_{p,2} \text{ corresponds to } \sin x.$$

For x, try the linear function

$$y_{p,1} = Ax + B.$$

For $\sin x$, the naive trial $A\cos x + B\sin x$ overlaps with y_h, so use

$$y_{p,2} = x(C\cos x + D\sin x).$$

The full substitution is somewhat long, but routine. After solving for coefficients, one finds

$$y_p = x - \frac{1}{2}x\cos x.$$

Thus the solution is

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x + x - \frac{1}{2}x\cos x.}$$

When the Method Does Not Apply The method of undetermined coefficients is powerful, but it does *not* apply to every forcing term. For example, it is generally not suitable for

$$\ln x, \qquad \tan x, \qquad \frac{1}{x}, \qquad e^{x^2}.$$

Why not? Because derivatives of these functions do not remain inside a finite-dimensional family of the same general type. In such cases we need more general methods, especially *variation of parameters*, which we will study later.

Problem 4.5.5. *Explain why the method of undetermined coefficients is not appropriate for solving the following differential equation:*

$$y'' + y = \ln x.$$

Solution. The method works only when repeated differentiation of the forcing term stays within a finite family of similar functions. For $\ln x$, the derivatives are

$$\frac{1}{x}, \quad -\frac{1}{x^2}, \quad \frac{2}{x^3}, \ldots$$

These do not cycle or remain in a fixed finite-dimensional family like polynomials, exponentials, or sine/cosine combinations. So there is no reasonable finite trial form with undetermined coefficients. A different method is needed.

A Longer Mixed Example

Example 4.5.7. *Solve the differential equation*

$$y'' - 2y' - 3y = 2e^x + x.$$

Solution. We begin with the homogeneous equation:

$$y'' - 2y' - 3y = 0.$$

The characteristic equation is

$$r^2 - 2r - 3 = 0,$$

which factors as

$$(r-3)(r+1) = 0.$$

So

$$y_h = c_1 e^{3x} + c_2 e^{-x}.$$

Now the forcing term is a sum:

$$2e^x + x.$$

Since the operator is linear, we look for

$$y_p = y_{p,1} + y_{p,2},$$

where

$$y_{p,1} \text{ matches } 2e^x, \qquad y_{p,2} \text{ matches } x.$$

For $2e^x$, try

$$y_{p,1} = Ae^x.$$

For x, try the linear function

$$y_{p,2} = Bx + C.$$

So our full trial function is

$$y_p = Ae^x + Bx + C.$$

Differentiate:

$$y_p' = Ae^x + B, \qquad y_p'' = Ae^x.$$

Substitute into the differential equation:

$$Ae^x - 2(Ae^x + B) - 3(Ae^x + Bx + C) = 2e^x + x.$$

Expand:

$$Ae^x - 2Ae^x - 2B - 3Ae^x - 3Bx - 3C = 2e^x + x.$$

Collect terms:

$$(-4A)e^x + (-3B)x + (-2B - 3C) = 2e^x + x.$$

Now match coefficients:

$$-4A = 2, \qquad -3B = 1, \qquad -2B - 3C = 0.$$

Thus

$$A = -\frac{1}{2}, \qquad B = -\frac{1}{3}.$$

Finally,

$$-2\left(-\frac{1}{3}\right) - 3C = 0 \quad \Longrightarrow \quad \frac{2}{3} - 3C = 0 \quad \Longrightarrow \quad C = \frac{2}{9}.$$

So the particular solution is

$$y_p = -\frac{1}{2}e^x - \frac{1}{3}x + \frac{2}{9}.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^x - \frac{1}{3}x + \frac{2}{9}.}$$

Practice Problems

Problem 4.5.6. *Solve the following differential equations by the method of undetermined coefficients.*

1. $y'' - y = 4$
2. $y'' + 3y' + 2y = e^{-x}$
3. $y'' + y = 2\cos x$
4. $y'' - 2y' + y = e^x$
5. $y'' + 4y = x$
6. $y'' - 3y' + 2y = e^x + x^2$
7. $y'' + 2y' + 5y = e^{-x}\cos 2x$

Solution.

1. Since the forcing term is constant, try $y_p = A$. Substitution gives $-A = 4$, so $A = -4$. Hence

$$y = c_1 e^x + c_2 e^{-x} - 4.$$

2. The homogeneous solution is
$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$
Since e^{-x} overlaps with y_h, try
$$y_p = Axe^{-x}.$$
Substitution gives $A = 1$, so
$$y = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x}.$$

3. Since $\cos x$ resonates with the homogeneous solution, try
$$y_p = x(A\cos x + B\sin x).$$
Substitution gives $A = 0$, $B = 1$, so
$$y = c_1 \cos x + c_2 \sin x + x \sin x.$$

4. The homogeneous equation has repeated root $r = 1$, so try
$$y_p = Ax^2 e^x.$$
Substitution gives $A = \frac{1}{2}$, hence
$$y = (c_1 + c_2 x)e^x + \frac{1}{2}x^2 e^x.$$

5. Try
$$y_p = Ax + B.$$
Substitution gives
$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x.$$

6. Use
$$y_p = Ae^x + Bx^2 + Cx + D.$$
Solve for the coefficients after substitution.

7. Since the forcing term is already of the form $e^{ax}\cos bx$, try
$$y_p = e^{-x}(A\cos 2x + B\sin 2x).$$
Here resonance occurs because the homogeneous roots are $-1 \pm 2i$, so multiply by the most basic linear term, x:
$$y_p = xe^{-x}(A\cos 2x + B\sin 2x).$$

Key Takeaway The method of undetermined coefficients is one of the most efficient tools for solving nonhomogeneous linear differential equations with constant coefficients, but it only works for certain families of forcing terms. When it applies, it is often far faster than more general techniques.

The heart of the method is not memorizing a table, but recognizing patterns:

- What family does $g(x)$ belong to?
- What trial form stays inside that family under differentiation?
- Does that trial form overlap with the homogeneous solution?
- If so, how many powers of x are needed to remove the overlap?

In the next section we develop a more general method that works even when undetermined coefficients fails.

4.6 Variation of Parameters

In the previous section we studied the method of undetermined coefficients. That method is extremely efficient when the forcing term belongs to a small family of functions whose derivatives remain in the same general class, such as polynomial, exponential, sine, and cosine. However, the method has an important limitation: it does not apply to arbitrary forcing terms.

For example, if the nonhomogeneous term is

$$g(x) = \ln x, \qquad g(x) = \tan x, \qquad g(x) = \frac{1}{x}, \qquad \text{or} \qquad g(x) = e^{x^2},$$

there is no obvious finite trial form whose derivatives remain in a manageable family. In such cases we need a more flexible method.

The method of *variation of parameters* provides exactly that flexibility. It works for general second-order linear differential equations, provided we already know a fundamental set of solutions of the corresponding homogeneous equation.

The Setting We consider a second-order linear nonhomogeneous equation in standard form:

$$y'' + p(x)y' + q(x)y = g(x).$$

Suppose that the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

has two linearly independent solutions $y_1(x)$ and $y_2(x)$. Then the general homogeneous solution is

$$y_h(x) = c_1y_1(x) + c_2y_2(x).$$

The key idea of variation of parameters is simple but powerful: instead of looking for a particular solution with constant coefficients, we allow the coefficients to vary with x.

Thus we seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where u_1 and u_2 are unknown functions to be determined.

This is the point where the method gets its name: the “parameters” c_1 and c_2 from the homogeneous solution are no longer treated as constants, but as functions.

Why an Extra Condition Is Needed At first sight, this may seem like a very natural guess. But it introduces two unknown functions, u_1 and u_2, and if we differentiate y_p directly, the resulting expressions become quite complicated. To simplify the system, we impose an auxiliary condition:

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0.$$

This condition is not forced by the differential equation itself; rather, it is chosen for convenience. It reduces the complexity of the derivatives and makes the method workable.

Under this assumption,

$$y_p = u_1y_1 + u_2y_2,$$

and differentiating gives

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Because of the auxiliary condition, the terms involving u_1' and u_2' cancel, so

$$y_p' = u_1y_1' + u_2y_2'.$$

Differentiating once more,

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Now substitute y_p, y_p', and y_p'' into the differential equation

$$y'' + p(x)y' + q(x)y = g(x).$$

We obtain

$$\big(u_1'y_1' + u_2'y_2'\big) + u_1\big(y_1'' + py_1' + qy_1\big) + u_2\big(y_2'' + py_2' + qy_2\big) = g(x).$$

But y_1 and y_2 are solutions of the homogeneous equation, so

$$y_1'' + py_1' + qy_1 = 0, \qquad y_2'' + py_2' + qy_2 = 0.$$

Therefore the equation simplifies to

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = g(x).$$

So the unknown functions u_1 and u_2 must satisfy the system

$$u_1'y_1 + u_2'y_2 = 0,$$

$$u_1'y_1' + u_2'y_2' = g(x).$$

This is now a 2×2 linear system for u_1' and u_2'.

The Wronskian Appears Naturally The coefficient matrix of this system is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix},$$

whose determinant is precisely the Wronskian

$$W\{y_1, y_2\}(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

Since y_1 and y_2 are linearly independent, the Wronskian is nonzero on the interval under consideration. Thus the system can be solved uniquely.

Using Cramer's rule, we obtain

$$u_1'(x) = -\frac{y_2(x)g(x)}{W\{y_1, y_2\}(x)}, \qquad u_2'(x) = \frac{y_1(x)g(x)}{W\{y_1, y_2\}(x)}.$$

Integrating,

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W\{y_1, y_2\}(x)}\,dx, \qquad u_2(x) = \int \frac{y_1(x)g(x)}{W\{y_1, y_2\}(x)}\,dx.$$

Substituting back into

$$y_p = u_1 y_1 + u_2 y_2,$$

we obtain a particular solution.

Theorem 4.6.1 (Variation of Parameters)**.** *Let y_1 and y_2 be linearly independent solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0.$$

Then a particular solution of

$$y'' + p(x)y' + q(x)y = g(x)$$

is given by

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W\{y_1, y_2\}(x)}\,dx, \qquad u_2(x) = \int \frac{y_1(x)g(x)}{W\{y_1, y_2\}(x)}\,dx.$$

Therefore the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

A First Example

Example 4.6.1. *Solve*

$$y'' + y = \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Solution. The associated homogeneous equation is

$$y'' + y = 0.$$

Its characteristic equation is

$$r^2 + 1 = 0,$$

so a fundamental set of solutions is

$$y_1(x) = \cos x, \qquad y_2(x) = \sin x.$$

Their Wronskian is

$$W\{y_1, y_2\}(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

So this is a pleasant case: the Wronskian is simply 1.

Now the forcing term is

$$g(x) = \tan x.$$

Variation of parameters gives

$$u_1(x) = -\int y_2(x)g(x)\,dx = -\int \sin x \tan x\,dx,$$

$$u_2(x) = \int y_1(x)g(x)\,dx = \int \cos x \tan x\,dx.$$

Let us simplify each integral before rushing into computation.

Since

$$\tan x = \frac{\sin x}{\cos x},$$

we have

$$u_1(x) = -\int \frac{\sin^2 x}{\cos x}\,dx.$$

That integrand does not look pleasant at first glance. A useful identity is

$$\sin^2 x = 1 - \cos^2 x.$$

So

$$\frac{\sin^2 x}{\cos x} = \frac{1-\cos^2 x}{\cos x} = \sec x - \cos x.$$

Hence

$$u_1(x) = -\int (\sec x - \cos x)\,dx = -\int \sec x\,dx + \int \cos x\,dx.$$

Thus

$$u_1(x) = -\ln|\sec x + \tan x| + \sin x.$$

Now compute u_2:

$$u_2(x) = \int \cos x \tan x\,dx = \int \sin x\,dx = -\cos x.$$

Therefore

$$y_p = u_1 y_1 + u_2 y_2 = \big(-\ln|\sec x + \tan x| + \sin x\big)\cos x + (-\cos x)\sin x.$$

Notice that the last two terms cancel:

$$\sin x \cos x - \cos x \sin x = 0.$$

So the particular solution is

$$y_p(x) = -\cos x\,\ln|\sec x + \tan x|.$$

Thus the general solution is the sum of the homogeneous and particular solutions:

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x - \cos x\,\ln|\sec x + \tan x|.}$$

This is an excellent example of why variation of parameters matters: the forcing term $\tan x$ is not accessible by undetermined coefficients, but variation of parameters handles it naturally.

A Polynomial Example

Example 4.6.2. *Solve*

$$y'' - y = x.$$

Solution. We could solve this using undetermined coefficients, but it is useful to see how variation of parameters reproduces the result.

The homogeneous equation is

$$y'' - y = 0.$$

Its characteristic equation is

$$r^2 - 1 = 0,$$

with roots

$$r = \pm 1.$$

Hence a fundamental set of solutions is

$$y_1 = e^x, \qquad y_2 = e^{-x}.$$

Compute the Wronskian:

$$W\{y_1, y_2\} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2.$$

Now the forcing term is

$$g(x) = x.$$

So u_1 and u_2 are

$$u_1(x) = -\int \frac{y_2 g}{W}\, dx = -\int \frac{e^{-x}x}{-2}\, dx = \frac{1}{2}\int xe^{-x}\, dx,$$

$$u_2(x) = \int \frac{y_1 g}{W}\, dx = \int \frac{e^{x}x}{-2}\, dx = -\frac{1}{2}\int xe^{x}\, dx.$$

At this point the method asks us to compute two integrals. So the real work is no longer guessing, but integration.

Using integration by parts:

$$\int xe^{-x}\, dx = -(x+1)e^{-x}, \qquad \int xe^{x}\, dx = (x-1)e^{x}.$$

Therefore

$$u_1(x) = -\frac{1}{2}(x+1)e^{-x}, \qquad u_2(x) = -\frac{1}{2}(x-1)e^{x}.$$

Now build the particular solution:

$$y_p = u_1 y_1 + u_2 y_2.$$

Substitute:

$$y_p = \left(-\frac{1}{2}(x+1)e^{-x}\right)e^x + \left(-\frac{1}{2}(x-1)e^{x}\right)e^{-x}.$$

The exponentials cancel:

$$y_p = -\frac{1}{2}(x+1) - \frac{1}{2}(x-1) = -x.$$

Thus

$$y_p = -x,$$

and the general solution is

$$\boxed{y(x) = c_1 e^x + c_2 e^{-x} - x.}$$

A Trigonometric Example

Example 4.6.3. *Solve*

$$y'' + 4y = \sec(2x).$$

Solution. The homogeneous equation is

$$y'' + 4y = 0.$$

Its characteristic equation is

$$r^2 + 4 = 0,$$

so the roots are

$$r = \pm 2i.$$

Thus we may take

$$y_1(x) = \cos 2x, \qquad y_2(x) = \sin 2x.$$

Now compute the Wronskian:

$$W\{y_1, y_2\} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2.$$

The forcing term is

$$g(x) = \sec(2x).$$

Therefore

$$u_1(x) = -\int \frac{y_2 g}{W}\,dx = -\frac{1}{2}\int \sin 2x \sec 2x\,dx,$$

$$u_2(x) = \int \frac{y_1 g}{W}\,dx = \frac{1}{2}\int \cos 2x \sec 2x\,dx.$$

Simplify:

$$\sin 2x \sec 2x = \tan 2x, \qquad \cos 2x \sec 2x = 1.$$

So we obtain

$$u_1(x) = -\frac{1}{2}\int \tan 2x\,dx, \qquad u_2(x) = \frac{1}{2}\int 1\,dx.$$

We know that

$$\int \tan 2x\,dx = -\frac{1}{2}\ln|\cos 2x|,$$

so we can determine both u_1 and u_2:

$$u_1(x) = \frac{1}{4}\ln|\cos 2x|.$$

$$u_2(x) = \frac{x}{2}.$$

Thus the particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{4} \cos 2x \ \ln|\cos 2x| + \frac{x}{2} \sin 2x.$$

Hence the general solution is

$$\boxed{y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \cos 2x \ \ln|\cos 2x| + \frac{x}{2} \sin 2x.}$$

A Conceptual Comparison with Undetermined Coefficients It is worth pausing to compare the two methods.

- **Undetermined coefficients** is usually faster when it applies, but it only works for special forcing terms.
- **Variation of parameters** is more general, but it usually requires more algebra and integration.

So in practice the question is not "Which method is better in the abstract?" but rather "Does undetermined coefficients apply here?" If yes, it is often the more efficient choice. If not, variation of parameters becomes essential.

Problem 4.6.1. *For each equation below, decide whether undetermined coefficients applies. If it does not, explain why variation of parameters is the more suitable method.*

1. $y'' + y = \ln x$
2. $y'' - 4y = e^{2x}$
3. $y'' + 9y = \tan x$
4. $y'' + y' = x^2 + e^x$

Solution.

1. Undetermined coefficients does not apply, because $\ln x$ does not belong to one of the standard finite families closed under differentiation. Variation of parameters is appropriate.
2. Undetermined coefficients applies, because e^{2x} is an exponential forcing term.
3. Undetermined coefficients does not apply, because $\tan x$ is not part of the standard admissible families. Variation of parameters is appropriate.
4. Undetermined coefficients applies, because both x^2 and e^x are admissible forcing terms, and the method extends to sums of such terms.

A More Systematic Derivation of the Formulas Because the variation of parameters formulas are easy to memorize but also easy to misuse, it is worth reviewing their origin one more time in compact form.

We seek

$$y_p = u_1 y_1 + u_2 y_2$$

with auxiliary condition

$$u_1' y_1 + u_2' y_2 = 0.$$

Then

$$u_1' y_1' + u_2' y_2' = g(x).$$

So u_1' and u_2' satisfy the matrix system

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

This perspective is useful because it shows that the method is fundamentally a linear algebra computation built on the fundamental solutions.

Problem 4.6.2. *Explain why the Wronskian must be nonzero in order for variation of parameters to work.*

Solution. To find u_1' and u_2', we must solve the linear system

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

A 2×2 linear system has a unique solution precisely when its determinant is nonzero. That determinant is the Wronskian

$$W\{y_1, y_2\} = y_1 y_2' - y_2 y_1'.$$

So if the Wronskian were zero, the system could not be solved uniquely for u_1' and u_2'. Since variation of parameters depends on solving that system, a nonzero Wronskian is essential.

A Longer Worked Example

Example 4.6.4. *Solve*

$$y'' - 2y' + y = \frac{e^x}{x}, \qquad x > 0.$$

Solution. This is a very instructive example. The forcing term $\frac{e^x}{x}$ is not of a standard undetermined coefficients type, so variation of parameters is the appropriate method.

First solve the homogeneous equation:

$$y'' - 2y' + y = 0.$$

Its characteristic equation is

$$r^2 - 2r + 1 = 0,$$

so factoring yields

$$(r-1)^2 = 0.$$

Therefore a fundamental set of homogeneous solutions is

$$y_1 = e^x, \qquad y_2 = xe^x.$$

Now compute the Wronskian:

$$W\{y_1, y_2\} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}.$$

Evaluate the determinant carefully:

$$W = e^x(e^x + xe^x) - xe^x(e^x) = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}.$$

The forcing term is

$$g(x) = \frac{e^x}{x}.$$

Thus

$$u_1(x) = -\int \frac{y_2 g}{W}\,dx = -\int \frac{(xe^x)(e^x/x)}{e^{2x}}\,dx.$$

Now simplify before integrating:

$$\frac{(xe^x)(e^x/x)}{e^{2x}} = 1.$$

So $u_1(x)$ is given by

$$u_1(x) = -\int 1\,dx = -x.$$

Next, we solve for $u_2(x)$:

$$u_2(x) = \int \frac{y_1 g}{W}\,dx = \int \frac{e^x(e^x/x)}{e^{2x}}\,dx = \int \frac{1}{x}\,dx = \ln x.$$

At this point, we form the particular solution:

$$y_p = u_1 y_1 + u_2 y_2 = (-x)e^x + (\ln x)(xe^x).$$

which becomes

$$y_p = -xe^x + xe^x \ln x.$$

Therefore the general solution is

$$\boxed{y(x) = c_1 e^x + c_2 xe^x - xe^x + xe^x \ln x, \qquad x > 0.}$$

This example illustrates exactly where variation of parameters shines: the forcing term is not one that invites a finite trial guess, but the method still works cleanly.

Practice Problems

Problem 4.6.3. *Use variation of parameters to solve the following differential equations.*

1. $y'' + y = \sec x$
2. $y'' - y = e^x$
3. $y'' + 4y = \tan 2x$
4. $y'' - 2y' + y = e^x \ln x$
5. $y'' + y = \csc x$

Solution.

1. For $y'' + y = \sec x$, use $y_1 = \cos x$, $y_2 = \sin x$, and $W = 1$. Then
$$u_1 = -\int \sin x \sec x \, dx, \qquad u_2 = \int \cos x \sec x \, dx.$$
Simplify and integrate to obtain the particular solution.

2. For $y'' - y = e^x$, the homogeneous solutions are e^x and e^{-x}, with Wronskian -2. You may compare the resulting answer with what undetermined coefficients would produce.

3. For $y'' + 4y = \tan 2x$, use $y_1 = \cos 2x$, $y_2 = \sin 2x$, and $W = 2$. Then compute
$$u_1 = -\frac{1}{2}\int \sin 2x \tan 2x \, dx, \qquad u_2 = \frac{1}{2}\int \cos 2x \tan 2x \, dx.$$

4. For $y'' - 2y' + y = e^x \ln x$, use the repeated-root basis $y_1 = e^x$, $y_2 = xe^x$. The Wronskian is e^{2x}.

5. For $y'' + y = \csc x$, use $y_1 = \cos x$, $y_2 = \sin x$, and $W = 1$. Then
$$u_1 = -\int \sin x \csc x \, dx, \qquad u_2 = \int \cos x \csc x \, dx.$$

Final Perspective The method of variation of parameters is one of the most important general techniques in the theory of linear differential equations. Its main strength is flexibility: unlike undetermined coefficients, it does not depend on the forcing term belonging to a special family.

Its weakness is computational. Even when the theory is clean, the resulting integrals may be complicated. But this tradeoff is exactly what makes the method valuable: it gives us a systematic path forward even when easier methods fail.

So the practical philosophy is this:

- Try undetermined coefficients first when the forcing term suggests it.
- Use variation of parameters when no reasonable trial form exists.

Together, these methods give us a substantial toolkit for solving a wide variety of nonhomogeneous linear differential equations.

5 Spectral Theory

Spectral theory studies linear operators by analyzing their eigenvalues and eigenvectors. The central idea is that many linear transformations can be understood by decomposing them into simpler, independent components. This decomposition reveals the "spectrum" of the operator−the set of scalars that encode its essential behavior.

5.1 Introduction to Eigenvalues and Eigenvectors

In earlier chapters we studied matrices as tools for solving systems of linear equations and describing linear transformations. In this chapter we investigate a deeper structural property of matrices. Certain vectors behave in an especially simple way when multiplied by a matrix: they retain their direction and are merely scaled by a constant factor. These vectors are called *eigenvectors*, and the scaling factors are called *eigenvalues*.

Eigenvalues and eigenvectors form the foundation of *spectral theory*. They allow us to understand the internal structure of linear transformations and play an essential role in differential equations, stability theory, physics, data science, and many other fields.

Eigenvalues and Eigenvectors

Definition 5.1.1. *Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v}$ is called an* eigenvector *of A if there exists a scalar λ such that*

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The scalar λ is called an eigenvalue *corresponding to the eigenvector $\mathbf{v}$.*

The condition $A\mathbf{v} = \lambda\mathbf{v}$ means that the matrix A transforms the vector $\mathbf{v}$ into a scalar multiple of itself.

- The vector $\mathbf{v}$ does not change direction.
- The magnitude is scaled by λ.

Geometrically, eigenvectors represent directions that are preserved by the linear transformation represented by A.

Remark. The zero vector $\mathbf{v} = \mathbf{0}$ is excluded from the definition because the equation

$$A\mathbf{0} = \lambda\mathbf{0}$$

holds for every matrix and every scalar λ, which would make the definition meaningless.

Problem 5.1.1. *Determine whether the vector*

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

If it is, find the corresponding eigenvalue.

Solution. To test whether $\mathbf{v}$ is an eigenvector, we compute $A\mathbf{v}$.

$$A\mathbf{v} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Now observe that

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus

$$A\mathbf{v} = 3\mathbf{v}.$$

Therefore the vector $\mathbf{v}$ is indeed an eigenvector, and the corresponding eigenvalue is

$$\boxed{\lambda = 3}.$$

Deriving the Eigenvalue Equation Starting from the definition

$$A\mathbf{v} = \lambda\mathbf{v},$$

we rewrite the equation as

$$A\mathbf{v} - \lambda\mathbf{v} = 0.$$

Factoring out $\mathbf{v}$ gives

$$(A - \lambda I)\mathbf{v} = 0.$$

This is a homogeneous linear system. For a nontrivial solution $\mathbf{v} \neq 0$ to exist, the matrix $A - \lambda I$ must be singular.

Definition 5.1.2. *The equation*

$$\det(A - \lambda I) = 0$$

is called the characteristic equation *of the matrix A.*

Definition 5.1.3. *The polynomial*

$$p(\lambda) = \det(A - \lambda I)$$

is called the characteristic polynomial *of A.*

The roots of the characteristic polynomial are precisely the eigenvalues of the matrix.

Example: Finding Eigenvalues

Problem 5.1.2. *Find the eigenvalues of the matrix*

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Solution. We begin by forming the matrix

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = 0.$$

Compute the determinant:

$$(4-\lambda)(3-\lambda) - 2.$$

Expanding gives

$$12 - 7\lambda + \lambda^2 - 2.$$

Thus

$$\lambda^2 - 7\lambda + 10 = 0.$$

Factoring,

$$(\lambda - 5)(\lambda - 2) = 0.$$

Therefore the eigenvalues are

$$\boxed{\lambda_1 = 5, \quad \lambda_2 = 2}.$$

Finding Eigenvectors Once an eigenvalue is known, the corresponding eigenvectors are obtained by solving

$$(A - \lambda I)\mathbf{v} = 0.$$

Problem 5.1.3. *Find the eigenvectors corresponding to the eigenvalues of the matrix*

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Solution. We already determined the eigenvalues

$$\lambda_1 = 5, \qquad \lambda_2 = 2.$$

Eigenvalue $\lambda = 5$
Compute

$$A - 5I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Solving $(A - 5I)\mathbf{v} = 0$ gives

$$v_1 = v_2.$$

Thus the eigenvectors have the form

$$\mathbf{v} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad c \neq 0.$$

Eigenvalue $\lambda = 2$
Compute

$$A - 2I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

Solving

$$2v_1 + v_2 = 0$$

gives

$$\mathbf{v} = c \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus the matrix has two eigenvector directions.

Eigenvalues of Triangular Matrices Certain matrices allow eigenvalues to be determined immediately.

Theorem 5.1.1. *The eigenvalues of a triangular matrix are precisely its diagonal entries.*

Problem 5.1.4. *Find the eigenvalues of the matrix*

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 7 \end{pmatrix}.$$

Solution. Because the matrix is upper triangular, the determinant of $A - \lambda I$ is simply the product of the diagonal entries:

$$(2 - \lambda)(5 - \lambda)(7 - \lambda) = 0.$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 5, 7}.$$

Repeated Eigenvalues Eigenvalues may appear multiple times as roots of the characteristic polynomial.

Definition 5.1.4. *The number of times an eigenvalue appears as a root of the characteristic polynomial is called its* algebraic multiplicity.

Problem 5.1.5. *Find the eigenvalues of the matrix*

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Solution. Compute

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}.$$

The determinant is

$$(3-\lambda)^2.$$

Thus the characteristic equation is

$$(3-\lambda)^2 = 0.$$

Therefore

$$\boxed{\lambda = 3}$$

with algebraic multiplicity 2.

Practice Problems

Problem 5.1.6. *Find the eigenvalues of the matrix*

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Solution. Compute

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}.$$

This equals

$$(1-\lambda)^2 - 4.$$

Setting the determinant to 0, we have

$$\lambda^2 - 2\lambda - 3 = 0.$$

Factoring,

$$(\lambda - 3)(\lambda + 1) = 0.$$

Hence the eigenvalues are

$$\boxed{\lambda = 3, -1}.$$

Problem 5.1.7. *Verify that the vector*

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector of

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

Find the corresponding eigenvalue.

Solution. Compute

$$A\mathbf{v} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$A\mathbf{v} = -1\mathbf{v}.$$

Therefore the eigenvalue is

$$\boxed{\lambda = -1}.$$

Eigenvalues and eigenvectors reveal the internal structure of linear transformations. In the next section we will see how they allow certain matrices to be rewritten in a much simpler form through the process of *matrix diagonalization*. This idea will become extremely important when solving systems of differential equations and understanding matrix powers.

5.2 Introduction to Eigenvalues and Eigenvectors

In earlier chapters we studied matrices as tools for solving systems of linear equations and describing linear transformations. In this chapter we investigate a deeper structural property of matrices. Certain vectors behave in an especially simple way when multiplied by a matrix: they retain their direction and are merely scaled by a constant factor. These vectors are called *eigenvectors*, and the scaling factors are called *eigenvalues*.

Eigenvalues and eigenvectors form the foundation of *spectral theory*. They allow us to understand the internal structure of linear transformations and play an essential role in differential equations, stability theory, physics, data science, and many other fields.

Eigenvalues and Eigenvectors

Definition 5.2.1. *Let A be an $n \times n$ matrix. A nonzero vector* $\mathbf{v}$ *is called an* eigenvector *of A if there exists a scalar λ such that*

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The scalar λ is called an eigenvalue *corresponding to the eigenvector* $\mathbf{v}$.

The condition $A\mathbf{v} = \lambda\mathbf{v}$ means that the matrix A transforms the vector $\mathbf{v}$ into a scalar multiple of itself.

- The vector $\mathbf{v}$ does not change direction.
- The magnitude is scaled by λ.

Geometrically, eigenvectors represent directions that are preserved by the linear transformation represented by A.

Remark. The vector $\mathbf{v} = \mathbf{0}$ is excluded from the definition because the equation

$$A\mathbf{0} = \lambda\mathbf{0}$$

holds for every matrix and every scalar λ, which would make the definition meaningless.

Problem 5.2.1. *Determine whether the vector*

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

If it is, find the corresponding eigenvalue.

Solution. To test whether $\mathbf{v}$ is an eigenvector, we compute $A\mathbf{v}$.

$$A\mathbf{v} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Now observe that

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Thus

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Therefore the vector $\mathbf{v}$ is indeed an eigenvector, and the corresponding eigenvalue is

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we rewrite the equation as

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Factoring out $\mathbf{v}$ gives

$$(A - \lambda I)\mathbf{v} = 0.$$

This is a homogeneous linear system. For a nontrivial solution $\mathbf{v} \neq 0$ to exist, the matrix $A - \lambda I$ must be singular, i.e. $\det(A - \lambda I) = 0$.

Definition 5.2.2. *The equation*

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The characteristic equation is

$$\det(A - \lambda I) = 0.$$

Compute the determinant:

$$(4-\lambda)(3-\lambda) - 2.$$

Expanding gives

$$12 - 7\lambda + \lambda^2 - 2.$$

Thus

$$\lambda^2 - 7\lambda + 10 = 0.$$

Factoring,

$$(\lambda - 5)(\lambda - 2) = 0.$$

Therefore the eigenvalues are

$$\boxed{\lambda_1 = 5, \quad \lambda_2 = 2}.$$

Finding Eigenvectors Once an eigenvalue is known, the corresponding eigenvectors are obtained by solving

$$(A - \lambda I)\mathbf{v} = 0.$$

Problem 5.2.3. *Find the eigenvectors corresponding to the eigenvalues of the matrix*

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Solution. We already determined the eigenvalues

$$\lambda_1 = 5, \qquad \lambda_2 = 2.$$

Eigenvalue $\lambda = 5$
Compute

$$A - 5I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Solving $(A - 5I)\mathbf{v} = 0$ gives

$$v_1 = v_2.$$

Thus the eigenvectors have the form

$$\mathbf{v} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad c \neq 0.$$

Eigenvalue $\lambda = 2$
Compute

$$A - 2I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

Solving

$$2v_1 + v_2 = 0$$

gives

$$\mathbf{v} = c \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

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$$A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 7 \end{pmatrix}.$$

Solution. Because the matrix is upper triangular, the determinant of $A - \lambda I$ is simply the product of the diagonal entries:

$$(2-\lambda)(5-\lambda)(7-\lambda) = 0.$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 5, 7}.$$

Repeated Eigenvalues Eigenvalues may appear multiple times as roots of the characteristic polynomial.

Definition 5.2.4. *The number of times an eigenvalue appears as a root of the characteristic polynomial is called its* algebraic multiplicity*.*

Problem 5.2.5. *Find the eigenvalues of the matrix*

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Solution. Compute

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}.$$

The determinant is

$$(3-\lambda)^2.$$

Thus the characteristic equation is

$$(3-\lambda)^2 = 0.$$

Therefore

$$\boxed{\lambda = 3}$$

with algebraic multiplicity 2.

Practice Problems

Problem 5.2.6. *Find the eigenvalues of the matrix*

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Solution. Compute

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}.$$

This equals

$$(1-\lambda)^2 - 4.$$

Thus

$$\lambda^2 - 2\lambda - 3 = 0.$$

Factoring,

$$(\lambda - 3)(\lambda + 1) = 0.$$

Hence

$$\boxed{\lambda = 3, -1}.$$

Problem 5.2.7. *Verify that the vector*

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector of

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

Find the corresponding eigenvalue.

Solution. Compute

$$A\mathbf{v} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$A\mathbf{v} = -1\mathbf{v}.$$

Therefore the eigenvalue is

$$\boxed{\lambda = -1}.$$

Eigenvalues and eigenvectors reveal the internal structure of linear transformations. In the next section we will see how they allow certain matrices to be rewritten in a much simpler form through the process of *matrix diagonalization*. This idea will become extremely important when solving systems of differential equations and understanding matrix powers.

5.3 Applications of Spectral Theory

In the previous sections we introduced eigenvalues, eigenvectors, and matrix diagonalization. We now explore how these concepts can be used to solve important mathematical problems. Spectral theory allows us to analyze powers of matrices, understand long–term behavior of dynamical systems, and simplify many computations involving linear transformations.

Throughout this section we will see that once a matrix is diagonalized, many seemingly complicated calculations become remarkably simple.

Computing Matrix Powers One of the most useful applications of diagonalization is computing powers of matrices.

Suppose a matrix A can be diagonalized so that

$$A = PDP^{-1},$$

where D is diagonal. Then

$$A^k = PD^kP^{-1}.$$

Since powers of diagonal matrices are easy to compute, this formula greatly simplifies the calculation of A^k.

Theorem 5.3.1. *If $A = PDP^{-1}$, where D is diagonal, then for any positive integer k,*

$$A^k = PD^kP^{-1}.$$

Proof. We verify this using induction.

First observe

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}.$$

Continuing this pattern gives

$$A^k = PD^kP^{-1}.$$

□

Problem 5.3.1. *Compute A^4 for*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Solution. Since A is already diagonal, the computation is immediate:

$$A^4 = \begin{pmatrix} 2^4 & 0 \\ 0 & 3^4 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}.$$

This example illustrates why diagonal matrices are easy to work with.

Example: Matrix Powers via Diagonalization

Problem 5.3.2. *Compute A^k for*

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Solution. From Section 5.1 we know the eigenvalues are

$$\lambda_1 = 5, \qquad \lambda_2 = 2.$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Construct

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

The diagonal matrix is

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Thus

$$A = PDP^{-1}.$$

Therefore

$$A^k = PD^kP^{-1}.$$

Since

$$D^k = \begin{pmatrix} 5^k & 0 \\ 0 & 2^k \end{pmatrix},$$

we obtain

$$A^k = P \begin{pmatrix} 5^k & 0 \\ 0 & 2^k \end{pmatrix} P^{-1}.$$

This expression provides an explicit formula for any power of A.

Discrete Dynamical Systems Eigenvalues also play an important role in analyzing discrete dynamical systems.

Definition 5.3.1. *A* discrete dynamical system *is a sequence of vectors defined by*

$$\mathbf{x}_{k+1} = A\mathbf{x}_k.$$

Starting from an initial vector $\mathbf{x}_0$*, the state after* k *steps is*

$$\mathbf{x}_k = A^k\mathbf{x}_0.$$

Thus understanding the behavior of $\mathbf{x}_k$ reduces to understanding the powers of A.

Long-Term Behavior Eigenvalues can determine whether solutions grow, decay, or oscillate around the system over time.

- If $|\lambda| > 1$, the corresponding component grows.
- If $|\lambda| < 1$, the component decays.
- If $|\lambda| = 1$, the component remains bounded.

Problem 5.3.3. *Consider the system*

$$\mathbf{x}_{k+1} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbf{x}_k.$$

Describe the long-term behavior of solutions.

Solution. The eigenvalues of the matrix are

$$\lambda_1 = 2, \qquad \lambda_2 = \tfrac{1}{2}.$$

Thus

$$A^k = \begin{pmatrix} 2^k & 0 \\ 0 & (1/2)^k \end{pmatrix}.$$

As $k \to \infty$,

$$2^k \to \infty, \qquad (1/2)^k \to 0.$$

Therefore the first component grows without bound while the second component decays toward zero.

Connection with Differential Equations Spectral theory is also essential in solving systems of differential equations.

Consider the system

$$\mathbf{x}' = A\mathbf{x}.$$

If A has eigenvalues λ_i with eigenvectors $\mathbf{v}_i$, then solutions take the form

$$\mathbf{x}(t) = e^{\lambda_i t}\mathbf{v}_i.$$

Thus eigenvalues determine the growth rates of solutions.

Problem 5.3.4. *Verify that*

$$\mathbf{x}(t) = e^{2t}\begin{pmatrix}1\\0\end{pmatrix}$$

is a solution of the system

$$\mathbf{x}' = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}\mathbf{x}.$$

Solution. First compute

$$\mathbf{x}'(t) = 2e^{2t}\begin{pmatrix}1\\0\end{pmatrix}.$$

Next compute

$$A\mathbf{x}(t) = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}e^{2t}\begin{pmatrix}1\\0\end{pmatrix} = 2e^{2t}\begin{pmatrix}1\\0\end{pmatrix}.$$

Thus

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

so the function satisfies the system.

Practice Problems

Problem 5.3.5. *Find A^k for*

$$A = \begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}.$$

Solution. Since the matrix is diagonal,

$$A^k = \begin{pmatrix}3^k & 0\\0 & 4^k\end{pmatrix}.$$

Problem 5.3.6. *Suppose a dynamical system satisfies*

$$\mathbf{x}_{k+1} = \begin{pmatrix}0.8 & 0\\0 & 0.5\end{pmatrix}\mathbf{x}_k.$$

What happens to $\mathbf{x}_k$ as $k \to \infty$?

Solution. The eigenvalues are

$$0.8, \quad 0.5.$$

Both eigenvalues have magnitude less than 1, so

$$A^k \to 0.$$

Therefore all solutions approach the zero vector as $k \to \infty$.

Spectral theory reveals the structure hidden inside matrices and linear transformations. By studying eigenvalues and eigenvectors we can predict long–term behavior, compute matrix powers efficiently, and solve systems of differential equations.

In the next section we will explore an important geometric property of eigenvectors: *orthogonality.* This will lead to powerful results for symmetric matrices and further deepen our understanding of spectral theory.

5.4 Orthogonality of Eigenvectors

In the previous sections we studied eigenvalues, eigenvectors, and matrix diagonalization. We now investigate an important geometric property that arises when matrices possess additional structure. In particular, eigenvectors corresponding to certain matrices turn out to be *orthogonal* to one another.

This phenomenon plays a central role in spectral theory and leads to one of the most powerful results in linear algebra: the *spectral theorem* for symmetric matrices.

Orthogonality of Vectors Before discussing eigenvectors, we briefly recall the notion of orthogonality.

Definition 5.4.1. *Two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *are said to be* orthogonal *if their dot product is zero:*

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Orthogonality corresponds geometrically to vectors meeting at a right angle.

Problem 5.4.1. *Determine whether the vectors*

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

are orthogonal.

Solution. Compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = 1(2) + 2(-1) = 2 - 2 = 0.$$

Since the dot product is zero, the vectors are orthogonal. $\therefore \boxed{\mathbf{u} \perp \mathbf{v}}$.

Symmetric Matrices Orthogonality of eigenvectors arises naturally when a matrix is symmetric.

Definition 5.4.2. *A matrix A is called* symmetric *if*

$$A^T = A.$$

For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

is symmetric because its transpose equals itself.

Symmetric matrices possess several remarkable properties, one of which concerns their eigenvectors.

Orthogonality of Eigenvectors

Theorem 5.4.1. *Let A be a symmetric matrix. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.*

Proof. Suppose

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \qquad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2.$$

Take the dot product of the first equation with $\mathbf{v}_2$:

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

Because A is symmetric,

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (A\mathbf{v}_2).$$

Thus

$$\mathbf{v}_1 \cdot (A\mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

Equating the two expressions gives

$$\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

Since $\lambda_1 \neq \lambda_2$, the only possibility is

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

Therefore the eigenvectors are orthogonal.

□

Example: Orthogonal Eigenvectors

Problem 5.4.2. *Find the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and verify that the eigenvectors are orthogonal.

Solution. First compute the characteristic equation.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 - 1$$

Thus

$$\lambda^2 - 4\lambda + 3 = 0.$$

Factoring,

$$(\lambda - 3)(\lambda - 1) = 0.$$

So the eigenvalues are

$$\lambda_1 = 3, \qquad \lambda_2 = 1.$$

Eigenvalue $\lambda = 3$
Solve

$$(A - 3I)\mathbf{v} = 0.$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

This gives

$$v_1 = v_2.$$

Thus

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvalue $\lambda = 1$

$$(A - I) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This gives

$$v_1 = -v_2.$$

Thus

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now compute the dot product:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1(1) + 1(-1) = 0.$$

Therefore the eigenvectors are orthogonal.

Orthogonal Diagonalization Orthogonality of eigenvectors leads to a particularly elegant form of diagonalization.

Definition 5.4.3. *A matrix A is said to be* orthogonally diagonalizable *if*

$$A = QDQ^T$$

where Q is an orthogonal matrix and D is diagonal.

Recall that a matrix Q is orthogonal if

$$Q^T Q = I.$$

Theorem 5.4.2 (Spectral Theorem)**.** *Every real symmetric matrix is orthogonally diagonalizable.*

This result is one of the central theorems of linear algebra and forms the basis of much of spectral theory.

Example: Orthogonal Diagonalization

Problem 5.4.3. *Orthogonally diagonalize*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Solution. From the previous example the eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalize them to obtain orthonormal vectors.

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Construct a matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The diagonal matrix is

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

So via the definition of a diagonal matrix, we construct

$$A = QDQ^T.$$

Practice Problems

Problem 5.4.4. *Determine whether the vectors*

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}3\\-6\\3\end{pmatrix}$$

are orthogonal.

Solution. Compute the dot product:

$$1(3) + 2(-6) + 3(3) = 3 - 12 + 9 = 0.$$

Thus the vectors are orthogonal.

Problem 5.4.5. *Find the eigenvalues of the symmetric matrix*

$$\begin{pmatrix}3 & 1\\1 & 3\end{pmatrix}.$$

Solution. Compute

$$\det(A - \lambda I) = (3 - \lambda)^2 - 1.$$

Thus

$$\lambda^2 - 6\lambda + 8 = 0.$$

Factoring,

$$(\lambda - 4)(\lambda - 2) = 0.$$

Therefore

$$\lambda_1 = 4, \quad \lambda_2 = 2.$$

Orthogonality of eigenvectors provides a powerful geometric perspective on linear transformations. For symmetric matrices, eigenvectors form orthogonal directions that reveal the internal structure of the transformation.

In the following chapters we will apply these ideas to systems of differential equations and other applications of spectral theory.

6 Linear Systems of Differential Equations

In earlier chapters, we studied single differential equations like

$$y' = ky \quad \text{or} \quad y'' + \omega^2 y = 0.$$

But in many real-world situations, several quantities change together and affect each other at the same time. For example, two interacting populations, two masses connected by springs, or voltage and current in an electrical circuit. These are naturally modeled by *systems* of differential equations rather than just one equation. In this chapter, we will learn how to write systems of differential equations, solve homogeneous systems using eigenvalues and eigenvectors, understand the behavior of planar systems, and delve into the matrix exponential that allows us to solve even non-homogeneous systems.

6.1 What Is a Linear System?

In previous chapters we studied differential equations involving a single unknown function. However, many real-world phenomena involve several quantities that evolve simultaneously and influence one another. For example, the motion of a mechanical system may depend on multiple coordinates, and interacting populations in biology often require several variables to describe their dynamics.

Such problems naturally lead to *systems of differential equations*. In this chapter we study systems in which the unknown functions appear linearly. These are called *linear systems of differential equations*. They form one of the most important connections between differential equations and linear algebra.

Systems of Differential Equations

Definition 6.1.1. *A* system of differential equations *is a collection of differential equations involving several unknown functions and their derivatives.*

For example, the pair of equations

$$\frac{dx}{dt} = x + 2y, \qquad \frac{dy}{dt} = 3x - y$$

forms a system of two differential equations for the functions $x(t)$ and $y(t)$.

Remark. Throughout this chapter the independent variable will typically be t, representing time, although any independent variable could be used.

Problem 6.1.1. *Verify that the functions*

$$x(t) = e^{2t}, \qquad y(t) = e^{2t}$$

satisfy the linear system

$$\frac{dx}{dt} = 2x, \qquad \frac{dy}{dt} = 2y.$$

Solution. First compute the derivatives.

$$\frac{dx}{dt} = 2e^{2t}, \qquad \frac{dy}{dt} = 2e^{2t}.$$

Now substitute into the right-hand sides of the equations.

$$2x = 2e^{2t}, \qquad 2y = 2e^{2t}.$$

Thus

$$\frac{dx}{dt} = 2x, \qquad \frac{dy}{dt} = 2y,$$

so both equations are satisfied. Therefore the given functions form a solution of the system. □

Linear Systems

Definition 6.1.2. *A system of differential equations is called* linear *if each equation is linear in the unknown functions and their derivatives.*

More precisely, a system of n first-order differential equations is linear if it can be written in the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t),$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t),$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t).$$

Here the functions $a_{ij}(t)$ and $f_i(t)$ are known.

Remark. A system is linear because:

- each unknown function appears only to the first power
- the unknown functions are not multiplied together
- no nonlinear functions (such as $\sin x$, x^2, etc.) appear

Problem 6.1.2. *Determine whether the following systems are linear.*

(a)

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = 2x - y$$

(b)

$$\frac{dx}{dt} = x^2 + y, \qquad \frac{dy}{dt} = x - y$$

Solution. *(a)* Each equation contains only first powers of x and y, and the variables are not multiplied together. Therefore the system is linear. □

(b) The term x^2 appears in the first equation. Because the unknown function is squared, the system is nonlinear. □

Matrix Form of Linear Systems Linear systems can be written compactly using matrices.

Consider the system

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy.$$

Define the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then the system can be written as

$$\mathbf{x}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}(t).$$

Definition 6.1.3. *A linear system of differential equations written in the form*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

is called the matrix form *of the system.*

Here

$$A(t)$$

is a matrix of coefficients and

$$\mathbf{x}(t)$$

is the vector of unknown functions.

Problem 6.1.3. *Write the following system in matrix form:*

$$\frac{dx}{dt} = 3x + 4y$$

$$\frac{dy}{dt} = -2x + y$$

Solution. Define

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The coefficient matrix is

$$A = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}.$$

Thus the system becomes

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

This compact representation allows us to apply techniques from linear algebra to study the system.

Homogeneous and Nonhomogeneous Systems

Definition 6.1.4. *A system of the form*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

is called homogeneous*.*

Definition 6.1.5. *A system of the form*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

is called nonhomogeneous*, as* $\mathbf{f}(t) \neq 0$*.*

Problem 6.1.4. *Classify the system*

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = x - y + e^t$$

Solution. The second equation contains the term e^t, which does not depend on x or y. Therefore the system can be written

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t).$$

Thus the system is nonhomogeneous. □

Existence and Uniqueness Just as with single differential equations, solutions of linear systems satisfy an existence and uniqueness theorem.

Theorem 6.1.1. *Suppose the functions $a_{ij}(t)$ and $f_i(t)$ are continuous on an interval I. Then for any initial condition*

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

there exists a unique solution of the system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

on the interval I.

This result guarantees that linear systems behave predictably: once the initial state is specified, the future evolution of the system is uniquely determined.

Practice Problems

Problem 6.1.5. *Write the system*

$$\frac{dx}{dt} = 2x - y$$

$$\frac{dy}{dt} = x + 3y$$

in matrix form.

Solution. Let

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The coefficient matrix is

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}.$$

Thus the system becomes

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

Problem 6.1.6. *Verify that*

$$x(t) = e^t, \qquad y(t) = 2e^t$$

is a solution of

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = 2x.$$

Solution. Compute the derivatives:

$$\frac{dx}{dt} = e^t, \qquad \frac{dy}{dt} = 2e^t.$$

Now evaluate the right-hand sides.

$$x = e^t, \qquad 2x = 2e^t.$$

Thus

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = 2x,$$

so the functions satisfy the system. □

Linear systems of differential equations provide a powerful framework for modeling interacting quantities. In the next section we will study homogeneous systems with constant coefficients and develop systematic methods for solving them using eigenvalues and eigenvectors.

6.2 Homogeneous Systems with Constant Coefficients

In the previous section we introduced linear systems of differential equations and expressed them in matrix form. We now study one of the most important classes of such systems:

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A is a constant matrix. These are called *homogeneous systems with constant coefficients.*

Remarkably, the theory developed in Chapter 5 allows us to solve these systems using eigenvalues and eigenvectors of the matrix A.

Vector Solutions Recall that a solution of the system

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

is a vector-valued function

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose derivative satisfies the equation.

Problem 6.2.1. *Verify that the vector-valued function*

$$\mathbf{x}(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a solution of the below linear system:

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x}.$$

Solution. First compute the derivative.

$$\mathbf{x}'(t) = 3e^{3t}\begin{pmatrix}1\\2\end{pmatrix}.$$

Now compute the matrix product:

$$A\mathbf{x}(t) = \begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}e^{3t}\begin{pmatrix}1\\2\end{pmatrix} = e^{3t}\begin{pmatrix}3\\6\end{pmatrix}.$$

This equals

$$3e^{3t}\begin{pmatrix}1\\2\end{pmatrix}.$$

Thus we have demonstrated

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

so the function satisfies the system. □

Eigenvalue Solutions A key observation is that solutions often take the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

Here λ is a scalar and $\mathbf{v}$ is a constant vector.
Substituting this expression into the system reveals an important relationship.

Theorem 6.2.1. *Suppose λ is an eigenvalue of A with eigenvector $\mathbf{v}$. Then*

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution of the system

$$\mathbf{x}' = A\mathbf{x}.$$

Proof. Compute the derivative:

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v}.$$

Now compute

$$A\mathbf{x}(t) = Ae^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}.$$

Since $\mathbf{v}$ is an eigenvector, we obtain

$$A\mathbf{v} = \lambda\mathbf{v}.$$

So we have

$$A\mathbf{x}(t) = \lambda e^{\lambda t}\mathbf{v}.$$

Therefore, we have shown

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

and the proof is complete.

□

General Solution If a matrix has n linearly independent eigenvectors, then the solutions generated by those eigenvectors form a complete set of solutions.

Theorem 6.2.2. *Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.*

Then the general solution of

$$\mathbf{x}' = A\mathbf{x}$$

is the linear combination below:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n.$$

Here $c_1, \ldots, c_n \in \mathbb{R}$ are arbitrary constants.

Example: Solving a System

Problem 6.2.2. *Solve the system*

$$\mathbf{x}' = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \mathbf{x}.$$

Solution. First compute the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix}.$$

$$= (4-\lambda)(3-\lambda) - 2$$

$$= \lambda^2 - 7\lambda + 10.$$

Factoring gives $(\lambda - 5)(\lambda - 2) = 0$, so the roots are the eigenvalues:

$$\lambda_1 = 5, \qquad \lambda_2 = 2.$$

Next we find the eigenvectors.
For eigenvalue $\lambda = 5$:

$$(A - 5I) = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Solving gives

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For eigenvalue $\lambda = 2$:

$$(A - 2I) = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

Solving gives

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus the solutions are

$$\mathbf{x}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

So the general solution is the linear combination

$$\boxed{\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}}$$

Initial Value Problems To determine a particular solution we apply initial conditions.

Problem 6.2.3. *Solve the system*

$$\mathbf{x}' = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \mathbf{x}$$

subject to the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution. The general solution is

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Evaluate at $t = 0$:

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix}.$$

This gives the system

$$\begin{cases} c_1 + c_2 = 2 \\ c_1 - 2c_2 = 1 \end{cases}$$

Solving yields the values

$$c_1 = \frac{5}{3}, \qquad c_2 = \frac{1}{3}.$$

Thus the particular solution is

$$\boxed{\mathbf{x}(t) = \frac{5}{3}e^{5t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}e^{2t}\begin{pmatrix}1\\-2\end{pmatrix}}.$$

Practice Problems

Problem 6.2.4. *Solve the system*

$$\mathbf{x}' = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\mathbf{x}.$$

Solution. The eigenvalues are 2 and 3 with eigenvectors

$$\begin{pmatrix}1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\end{pmatrix}.$$

Thus the general solution to the system is

$$\boxed{\mathbf{x}(t) = c_1e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2e^{3t}\begin{pmatrix}0\\1\end{pmatrix}}.$$

Problem 6.2.5. *Find the eigenvalues of*

$$A = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$$

and write the corresponding solutions of $\mathbf{x}' = A\mathbf{x}$.

Solution. The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0.$$

This factors into $(\lambda - 3)(\lambda + 1)$ and the roots of the polynomial are the eigenvalues, so we get

$$\boxed{\lambda = 3, \quad \lambda = -1.}$$

Each eigenvalue produces a solution of the form

$$e^{\lambda t}\mathbf{v}.$$

Therefore the system has two independent exponential solutions:

$$\boxed{\mathbf{x}_1(t) = e^{3t}\begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix}2\\1\end{pmatrix}}$$

Homogeneous systems with constant coefficients can therefore be solved systematically using eigenvalues and eigenvectors. In the next section we will examine the *phase plane* and use geometric methods to analyze the qualitative behavior of solutions.

6.3 Phase Plane and Qualitative Behavior

In the previous section we developed methods for solving systems of differential equations using eigenvalues and eigenvectors. However, in many applications we are less interested in obtaining exact formulas for solutions and more interested in understanding the *qualitative behavior* of solutions.

For systems of two differential equations, this qualitative analysis can often be performed by examining the behavior of solution trajectories in the *phase plane*. The phase plane provides a geometric representation of how solutions evolve over time.

The Phase Plane Consider a system of two differential equations

$$\mathbf{x}' = A\mathbf{x},$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Each solution of the system defines a curve

$$(x(t), y(t))$$

in the xy-plane.

Definition 6.3.1. *The* phase plane *is the plane whose coordinates represent the variables of the system. A solution curve in this plane is called a* trajectory *or* phase path*.*

Thus instead of studying $x(t)$ and $y(t)$ separately, we analyze how the point $(x(t), y(t))$ moves through the plane.

Equilibrium Points A particularly important feature of dynamical systems is the presence of equilibrium points.

Definition 6.3.2. *A point* $\mathbf{x}_0$ *is called an* equilibrium point *of the system*

$$\mathbf{x}' = A\mathbf{x}$$

if we have

$$A\mathbf{x}_0 = 0.$$

At an equilibrium point the derivative is zero, meaning the system remains at rest if it starts there.

For homogeneous systems of the form

$$\mathbf{x}' = A\mathbf{x},$$

the origin is always an equilibrium point.

Problem 6.3.1. *Show that the origin is an equilibrium point of the system*

$$\frac{dx}{dt} = 3x - 2y, \qquad \frac{dy}{dt} = x + y.$$

Solution. Substitute $x = 0$ and $y = 0$ into the system.

$$\frac{dx}{dt} = 3(0) - 2(0) = 0,$$

$$\frac{dy}{dt} = 0 + 0 = 0.$$

Thus $\mathbf{x}'$ is simply the zero vector, i.e.

$$\mathbf{x}' = \mathbf{0}.$$

Therefore the origin is an equilibrium point. □

Direction Fields To understand how solutions move through the phase plane, we examine the direction of motion at each point.

At a point (x, y), the system determines a vector

$$\mathbf{x}' = A\mathbf{x}.$$

This vector indicates the instantaneous direction of the trajectory passing through that point.

Definition 6.3.3. *A* direction field *(or* vector field*) assigns to each point (x, y) the vector given by the differential equation.*

Plotting these vectors provides a picture of how trajectories behave throughout the phase plane.

Eigenvalues and Qualitative Behavior For linear systems

$$\mathbf{x}' = A\mathbf{x},$$

the qualitative behavior near the equilibrium point depends on the eigenvalues of the matrix A.

Let λ_1 and λ_2 be the eigenvalues of A. Several distinct cases arise.

Case 1: Real Eigenvalues of Opposite Sign (Saddle Point) If the eigenvalues satisfy the constraints

$$\lambda_1 > 0, \qquad \lambda_2 < 0,$$

the origin is called a *saddle point.*

In this case some trajectories approach the origin while others move away from it.

Problem 6.3.2. *Consider the linear system*

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x}.$$

Classify the equilibrium point.

Solution. The eigenvalues are the diagonal entries:

$$\lambda_1 = 2, \qquad \lambda_2 = -1.$$

One eigenvalue is positive and the other is negative. Therefore the equilibrium point is a **saddle point**.

Case 2: Both Eigenvalues Negative (Stable Node) If the eigenvalues satisfy the constraints

$$\lambda_1 < 0, \qquad \lambda_2 < 0,$$

then all trajectories approach the origin as $t \to \infty$.
The origin is called a *stable node*.

Problem 6.3.3. *Classify the equilibrium point for*

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}.$$

Solution. The eigenvalues are

$$\lambda_1 = -2, \qquad \lambda_2 = -3.$$

Both eigenvalues are negative, so solutions decay toward the origin.
Thus the equilibrium point is a **stable node**.

Case 3: Both Eigenvalues Positive (Unstable Node) If the eigenvalues satisfy the constraints

$$\lambda_1 > 0, \qquad \lambda_2 > 0,$$

then all nontrivial solutions grow away from the origin.
The origin is called an *unstable node*.

Problem 6.3.4. *Classify the equilibrium point of*

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}.$$

Solution. The eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 2.$$

Both eigenvalues are positive, so solutions grow exponentially away from the origin.
Thus the equilibrium point is an **unstable node**.

Complex Eigenvalues If the eigenvalues are complex conjugates

$$\lambda = a \pm bi,$$

the trajectories spiral around the equilibrium point. Specifically,

- If $a < 0$, trajectories spiral inward.
- If $a > 0$, trajectories spiral outward.

Problem 6.3.5. *Consider the system*

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

Describe the qualitative behavior of solutions.

Solution. The characteristic equation is

$$\lambda^2 + 1 = 0.$$

Thus the eigenvalues are

$$\lambda = \pm i.$$

The eigenvalues are purely imaginary, meaning solutions rotate around the origin.
Therefore trajectories form closed curves surrounding the origin. □

Phase Portraits The collection of all trajectories of a system in the phase plane is called its *phase portrait*.

Phase portraits provide a global picture of system behavior without requiring explicit formulas for solutions.

Here are the typical equilibrium classifications:

- saddle point
- stable node
- unstable node
- spiral point
- center

These patterns arise repeatedly in many dynamical systems.

Practice Problems

Problem 6.3.6. *Classify the equilibrium point of the system*

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x}.$$

Solution. The eigenvalues are

$$\lambda = 3, -2.$$

Since the eigenvalues have opposite signs, the equilibrium point is a **saddle point**.

Problem 6.3.7. *Classify the equilibrium point of the system*

$$\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x}.$$

Solution. The eigenvalues are

$$\lambda = -1, -4.$$

Both eigenvalues are negative, so trajectories approach the origin.
Thus the equilibrium point is a **stable node**.

Phase plane analysis provides a powerful geometric perspective on systems of differential equations. By examining eigenvalues of the coefficient matrix, we can predict the qualitative behavior of solutions without solving the system explicitly.

In the next section we will extend these ideas to *nonhomogeneous systems* and introduce the important concept of the *matrix exponential*.

6.4 Nonhomogeneous Systems and the Matrix Exponential

In the previous sections we studied homogeneous systems of differential equations of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A is a constant matrix. We now extend our analysis to the more general system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t),$$

which is called a *nonhomogeneous linear system*. To develop a systematic solution method, we introduce the important concept of the *matrix exponential*.

The Matrix Exponential Recall from Calculus II that the exponential function can be defined by the power series

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

This idea extends naturally to matrices.

Definition 6.4.1. *Let A be an $n \times n$ matrix. The* matrix exponential *is defined by the infinite series*

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

This infinite series converges for all matrices A and all real values of t.

Properties of the Matrix Exponential The matrix exponential shares several properties with the ordinary exponential function.

Theorem 6.4.1. *If A is a constant matrix, then the matrix exponential satisfies*

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Idea of the Proof. Differentiate the power series term by term:

$$\frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \cdots\right) = A + A^2t + \frac{A^3t^2}{2!} + \cdots$$

Factoring out A gives Ae^{At}, so the proof is complete. □

Solution of Homogeneous Systems The matrix exponential provides an elegant expression for solutions of homogeneous systems.

Theorem 6.4.2. *The solution of the system*

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

with initial condition

$$\mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

Thus the matrix exponential plays the same role for systems of differential equations that the scalar exponential plays for first-order equations.

Problem 6.4.1. *Verify that the vector function*

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

satisfies the linear system

$$\mathbf{x}' = A\mathbf{x}.$$

Solution. Differentiate the proposed solution:

$$\mathbf{x}'(t) = \frac{d}{dt}(e^{At}\mathbf{x}_0).$$

Since $\mathbf{x}_0$ is constant,

$$\mathbf{x}'(t) = (Ae^{At})\mathbf{x}_0.$$

Rearranging yields

$$\mathbf{x}'(t) = A(e^{At}\mathbf{x}_0).$$

However, we know that

$$e^{At}\mathbf{x}_0 = \mathbf{x}(t).$$

Therefore

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

so the formula satisfies the system. □

Computing the Matrix Exponential If the matrix A is diagonalizable, computing the matrix power e^{At} becomes much easier.

Suppose

$$A = PDP^{-1},$$

where D is diagonal.

Then the matrix exponential is

$$e^{At} = Pe^{Dt}P^{-1}.$$

Problem 6.4.2. *Compute e^{At} for*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Solution. Since the matrix is diagonal,

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}.$$

Thus each diagonal entry simply becomes the exponential of that entry multiplied by the parameter variable t.

Nonhomogeneous Systems We now return to the general system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t).$$

To solve this system we use the method of *variation of parameters.*

Theorem 6.4.3. *The general solution of the nonhomogeneous system*

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

is the vector-valued function

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-s)}\mathbf{f}(s)\,ds.$$

The first term represents the solution of the homogeneous system, while the integral accounts for the forcing term.

Example: Nonhomogeneous System

Problem 6.4.3. *Solve the system*

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

Solution. First compute

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

The homogeneous solution is

$$\mathbf{x}_h(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}.$$

To find a particular solution we observe that the forcing term affects only the first equation. Solving the resulting scalar equation yields

$$x_1(t) = te^t.$$

The second equation remains homogeneous:

$$x_2(t) = c_2 e^{2t}.$$

Thus the solution of the system is

$$\boxed{\mathbf{x}(t) = \begin{pmatrix} c_1 e^t + te^t \\ c_2 e^{2t} \end{pmatrix}}.$$

Practice Problems

Problem 6.4.4. *Compute e^{At} for*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Solution. Since A is diagonal (you can verify this yourself), the exponential is

$$\boxed{e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}}.$$

Problem 6.4.5. *Solve the homogeneous system*

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

Solution. The eigenvalues are 3 and 1. Therefore the solution is

$$\boxed{\mathbf{x}(t) = \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{t} \end{pmatrix}}.$$

The matrix exponential provides a powerful and unified method for solving linear systems of differential equations. It connects the theory of differential equations with the spectral theory of matrices developed in Chapter 5.

This concludes our study of linear systems of differential equations. In the next chapter we will explore further techniques for solving differential equations, including Laplace transforms and their applications.

7 Laplace Transform and Methods

The Laplace transform can also be used to solve differential equations and reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. This method has powerful applications in various fields of engineering.

7.1 Laplace Transforms: In Essence

In this section, we will introduce a very efficient method for solving a multitude of ODE problems. This is a tool that allows us to take a differential equation and convert it into an algebraic equation. If there exists a solution to algebraic equation, then we apply the inverse of the transform to obtain said solution. This method is called the **Laplace transform**. It also has applications in circuit analysis, nuclear magnetic radiation spectroscopy, signal processing and systems, etc.

The Laplace transform also gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. Consider the standard equation used to model mechanical vibrations in a harmonic oscillator

$$m\frac{d^2x}{dt^2} = c\frac{dx}{dt} + kx = f(t)$$

From a physics point of view, t represents time and $f(t)$ an incoming signal. The Laplace transform of $f(t)$ will convert this differential equation into a simpler algebraic equation, containing a new independent variable s to represent frequency.

We express the Laplace transform of $f(t)$ as $\mathcal{L}\{f(t)\} = F(s)$. By convention, we write lowercase letters for functions in the time domain and uppercase letters for functions in the frequency domain. The same letter is used to denote that one function is the Laplace transform of the other.

Definition 7.1.1. *For a function $f(t)$, its **Laplace transform** is defined as*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

In this transform, we only consider $t \geq 0$. Of course, if t is assumed to be time, then the convention of $t \geq 0$ is trivial. Otherwise, we are nevertheless interested in finding what happens in the future in the context of the Laplace transform.

Problem 7.1.1. *Find the Laplace transform of the following functions.*
(a) $f(t) = 1$
(b) $f(t) = e^{-at}$
(c) $f(t) = \frac{t}{2}$

Solution to part a: The Laplace transform of $f(t) = 1$ is $\mathcal{L}(1)$. Applying Definition 7.1.1, we have

$$\mathcal{L}(1) = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} = \lim_{h\to\infty}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{h} = \lim_{h\to\infty}\left(\frac{e^{-sh}}{-s} - \frac{1}{-s}\right) = \boxed{\frac{1}{s}}$$

Note that the improper integral exists only if $s > 0$, so $\mathcal{L}(1)$ is defined only in the case where $s > 0$.

Solution to part b: If $f(t) = e^{-at}$, then $\mathcal{L}(e^{-at})$ is given by

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st}\left(e^{-at}\right)\,dt = \int_0^\infty e^{-(s+a)t}\,dt = \left[\frac{e^{-(s+a)t}}{-s(s+a)}\right]_{t=0}^{\infty} = \boxed{\frac{1}{s+a}}$$

Clearly, limit exists if and only if $s + a > 0$. So the transform $\mathcal{L}(e^{-at})$ is defined only for $s + a > 0$.

Solution to part c: We'll need to use integration by parts to solve this part of the problem. It is always important to stay strong with the basics from Calculus I and II. The Laplace transform is $\mathcal{L}\left(\frac{t}{2}\right)$, given by

$$\begin{aligned}
\mathcal{L}\left(\frac{t}{2}\right) &= \int_0^\infty e^{-st}\left(\frac{t}{2}\right)\,dt \\
&= \frac{1}{2}\int_0^\infty te^{-st}\,dt \\
&= \frac{1}{2}\left(\left[\frac{-te^{-st}}{s}\right]_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right) \\
&= \frac{1}{2}\left(0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty}\right) = \boxed{\frac{1}{2s^2}}
\end{aligned}$$

Again, this limit only exists for $s > 0$.

Via similar procedures we can compute the Laplace transform for a number of elementary functions. The below table lists various basic transforms.

Since the transform is defined by an integral, we can use properties of linearity. Suppose C is a

$f(t)$	$\mathcal{L}\{f(t)\}$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^3	$\frac{6}{s^4}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$

constant. From the table, we have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} Cf(t)\, dt = C \int_0^\infty e^{-st} f(t)\, dt = C\mathcal{L}\{f(t)\}$$

Essentially, we can "pull" the constant outside of the transform. Similarly, Laplace transforms are governed by linearity.

Theorem 7.1.1. *Suppose $A, B, C \in \mathbb{R}$ are constants. Then we have*

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$$

as well as

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}$$

Proof. . We wish to demonstrate linearity via addition and scalar multiplication, i.e. show that $\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$.

We proceed by applying the definition of the Laplace transform.

$$\begin{aligned}
\mathcal{L}\{Af(t) + Bg(t)\} &= \int_0^\infty (Af(t) + Bg(t))e^{-st}\, dt \\
&= A\int_0^\infty f(t)e^{-st}\, dt + B\int_0^\infty g(t)e^{-st}\, dt \\
&= A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}
\end{aligned}$$

This completes the first part of the proof.

For the second part, consider the Laplace transform of $Cf(t)$, where C is a constant:

$$\begin{aligned}
\mathcal{L}\{Cf(t)\} &= \int_0^\infty Cf(t)e^{-st}\, dt \\
&= C\int_0^\infty f(t)e^{-st}\, dt \\
&= C\mathcal{L}\{f(t)\}
\end{aligned}$$

Thus, we have shown that $\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}$. Therefore, the Laplace transform is linear.

□

The property of linearity along with the table allows us to easily compute Laplace transforms of a wide variety of functions. Be careful about one thing, however. It's a very common mistake to think that the Laplace transform of a product of functions is equivalent to the product of the individual transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

Also, not all functions have a Laplace transform, e.g. $f(t) = \frac{1}{t}$ does not have a transform because the corresponding improper integral diverges for all values of s. This applies similarly for $\tan t$ and e^{t^2} functions.

The Existence and Uniqueness Theorems Let's ask yourselves: when does a Laplace transform generally exist? We start with functions of the exponential order. The function $f(t)$ is of exponential order as $t \to \infty$ if

$$|f(t)| \leq Me^{ct}$$

for some M and c for all $t > t_0$ for some t_0. The simplest way to check this condition is to attempt to compute the limit

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}} = L$$

If the limit exists and is finite (in most cases $L = 0$), then $f(t)$ is of exponential order.

Problem 7.1.2. *Show that $f(t) = t^n$ is exponential order for any n.*

Solution. The Laplace transform is $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\}$, which is given by the integral

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\, dt$$

To compute this, we will need to use integration by parts. We set $u = t^n$, so $du = nt^{n-1}\, dt$ and if $dv = e^{-st}\, dt$, we have $v = -\frac{1}{s}e^{-st}$.

$$\begin{aligned} \mathcal{L}\{t^n\} &= \int_0^\infty t^n e^{-st}\, dt \\ &= \left[-\frac{1}{s}t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\, dt \end{aligned}$$

Now, we evaluate the boundary term:

$$\lim_{t\to\infty}\left(-\frac{1}{s}t^n e^{-st}\right) = 0 \quad \text{(since } e^{-st} \text{ decays faster than } t^n \text{ grows)}$$

At $t = 0$, the term is also zero. Hence, the boundary term vanishes, and we are left with:

$$\mathcal{L}\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\, dt$$

We now repeat the process for the remaining integral $\int_0^\infty t^{n-1}e^{-st}\, dt$. We perform integration by parts again, setting $u = t^{n-1}$, so $du = (n-1)t^{n-2}\, dt$, $dv = e^{-st}\, dt$, therefore $v = -\frac{1}{s}e^{-st}$. This yields

$$\mathcal{L}\{t^n\} = \frac{n}{s}\left[-\frac{1}{s}t^{n-1}e^{-st}\right]_0^\infty + \frac{n(n-1)}{s^2}\int_0^\infty t^{n-2}e^{-st}\, dt$$

Again, the boundary term vanishes, and we are left with:

$$\mathcal{L}\{t^n\} = \frac{n(n-1)}{s^2}\int_0^\infty t^{n-2}e^{-st}\, dt$$

If we continue applying integration by parts n times, we eventually reach

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Thus, the Laplace transform $\mathcal{L}\{t^n\}$ exists for $\Re(s) > 0$, and the above result indicates that t^n grows at most as e^{ct} for some constant $c > 0$. Therefore, t^n is of exponential order for all n. □

For exponential order functions, the Laplace transform is governed by two significant rules: the existence and uniqueness theorems.

Theorem 7.1.2. *Suppose $f(t)$ is continuous for all t and is exponential order for a constant value c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all values $s > c$.*

Why does this work? If $f(t)$ is of exponential order, then $|f(t)| \leq Me^{ct}$ for all values $t > 0$ (we make an assumption that $t_0 = 0$). If $s > c$, we have $c - s < 0$. The improper integral which defines the Laplace transform $\mathcal{L}\{f(t)\}$ exists if and only if the following integral also exists.

$$\int_0^\infty e^{-st}\left(Me^{ct}\right)\,dt = M\int_0^\infty e^{(c-s)t}\,dt = M\left[\frac{e^{(c-s)t}}{c-s}\right]_{t=0}^{\infty} = \frac{M}{c-s}$$

Of course, this transform can definitely involve functions that are not of exponential order, but in the grand scheme of things, such functions are not very important to us. Before moving onto uniqueness, we recognize that the Laplace transform of exponential order functions decays at infinity, i.e.

$$\lim_{s\to\infty} F(s) = 0$$

Theorem 7.1.3. *Let $f(t)$ and $g(t)$ both be continuous functions of exponential order. There exists a constant C such that $F(s) = G(s)$ for all values $s > C$. So, $f(t) = g(t)$, i.e. the functions are identical for all $t \geq 0$.*

The existence and uniqueness theorems hold for piecewise-continuous functions too. These are functions that are continuous pretty much everywhere except for a set few discrete points, where they may "jump." However, the uniqueness theorem does not observe function values at the discontinuities, so we can only conclude that $F(s) = G(s)$ outside such cases.

Inverse Transforms We already know that a Laplace transform converts a differential equation into an algebraic equation. Now, when we solve the algebraic equation, we are in the frequency domain. So we want to get back to the time domain. In particular, if we have a function $F(s)$, we want to find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. We must know that the function is unique. Thankfully, the uniqueness theorem is our best friend here.

Definition 7.1.2. *If $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$, the* ***inverse Laplace transform*** *is defined as*

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

We could integrate to find the inverse but this requires an application of complex numbers and path integrals. It's more sufficient to use the table we already have.

Problem 7.1.3. *Find the inverse Laplace transform of $F(s) = \frac{1}{s+1}$.*

Solution. Looking at the table, we find that $\mathcal{L}^{-1}\{\frac{1}{s+1}\} = \boxed{e^{-t}}$.

Notice that the as the Laplace transform is linear, the inverse transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}$$

Let's see how linearity of inverse transforms can be applied.

Problem 7.1.4. *Find the inverse Laplace transform of*

$$F(s) = \frac{s^2 + s + 1}{s^3 + s}$$

Solution. We will have to use partial fraction decomposition to write F in a form where we can apply the Laplace transform table. The denominator in factored form is $s(s^2+1)$ so rewriting gives

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

If we equate the numerators after setting the right hand side over a common denominator, we have $A(s^2 + 1) + s(Bs + C) = s^2 + s + 1$. If we expand and match the proper coefficients, we obtain $A + B = 1$, $C = 1$, $A = 1$, and so B must be 0. In other words, we have

$$F(s) = \frac{s^2 + s + 1}{s^3 + s} = \frac{1}{s} + \frac{1}{s^2 + 1}$$

Applying the linearity of the inverse Laplace transform, we have

$$\mathcal{L}^{-1}\{\frac{s^2 + s + 1}{s^3 + s}\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \boxed{1 + \sin t}$$

Shifting Property of Laplace Transforms Finally, a useful property in Laplace transforms is the **shifting property**, also known as the *first shifting property*

$$\mathcal{L}\{e^{-at}f(t)\} = F(s + a)$$

where $F(s)$ is the Laplace transform of $f(t)$ and $a \in \mathbb{R}$ is a constant.

In most cases, the shifting property is used when the denominator is a more complicated quadratic that can arise with the method of partial fractions. We can complete the square and write such quadratics in the form $(s + a)^2 + b$ and then apply the shifting property.

Problem 7.1.5. *Find the inverse Laplace transform of*

$$F(s) = \frac{1}{s^2 + 4s + 8}$$

Solution. First, we complete the square to make the denominator $(s+2)^2+4$. Then

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$$

Applying the shifting property, we find

$$\mathcal{L}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}\left\{\frac{1}{(s+2)^2+4}\right\} = \boxed{\frac{1}{2}e^{-2t}\sin(2t)}$$

In general, our goal is to apply the Laplace transform to rational functions, i.e. functions of the form

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomial functions in terms of s. For functions that we consider, the Laplace transform approaches 0 as $s \to \infty$. In order for that to happen, the degree of $F(s)$ must be smaller than that of $G(s)$. These functions are called *proper* rational functions because we can always use partial fraction decomposition. Obviously, we need to be able to factor the denominator into linear and quadratic terms, which involves root-finding.

7.2 Transforms of Derivatives and Ordinary Differential Equations

Laplace Transforms of Derivatives Let's consider how the Laplace transform is actually used for ODEs. The first case is to find the transform for a function that is a derivative. If $g(t)$ is a differentiable function of exponential order, then $\exists M\, \exists c$ such that $|g(t)| \leq Me^{ct}$. So, it is implied that $\mathcal{L}\{g(t)\}$ exists, and $\lim_{t\to\infty} e^{-st}g(t) = 0$, $\forall s > c$. Then

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt = \left[e^{-st}g(t)\right]_{t=0}^{\infty} - \int_0^\infty (-s)e^{-st}g(t)\,dt = -g(0) + s\mathcal{L}\{g(t)\}$$

Remark. The symbols $\exists$ and $\forall$ are called *existential* and *universal* quantifiers, respectively, and they directly translate to "there exist some values" and "for all values."

Anyways, let's go back to the integral shown above. This procedure can be repeated for higher derivatives. The results are shown in the table below. The same can be applied for piecewise-smooth functions, i.e. piecewise continuous functions with piecewise continuous first derivatives. There isn't much importance lying in the fact that the function is of exponential order except the fact that all these limits exist.

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Using Laplace Transforms to Solve ODEs Notice here that differentiating n number of times turns into multiplication by s when the Laplace transform is applied. Let's see a practical application of this algebraic property.

Problem 7.2.1. *Find the general solution to the differential equation*

$$x''(t) + x(t) = \cos(2t)$$

with initial conditions $x(0) = 0$ *and* $x'(0) = 1$*.*

Solution. Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\}$$

$$s^2X(s) - sx(0) + x'(0) + X(s) = \frac{s}{s^2+4}$$

We plug in the initial conditions now−to streamline the computations−yielding

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

Solving the equation for $X(s)$, we have

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}$$

Using partial fraction decomposition (this exercise is left to you), we write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}$$

Finally, we take the inverse Laplace transform of $X(s)$ to get the general solution $x(t)$.

$$\boxed{x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t)}$$

There is a systematic process to solve any linear constant-coefficient differential equation. First, we take an ODE in the time domain t. Then we apply the Laplace transform to transform the equation into an algebraic, non-differential function in the frequency domain. Here, $x(t)$, $x'(t)$, $x''(t)$, and so on, get converted to $X(s)$, $sX(s) - x(0)$, $s^2X(s) - sx(0) - x'(0)$, and so on. We solve this algebraic equation for $X(s)$. Finally, we apply the inverse Laplace transform, if it exists, to find the general solution $x(t)$.

Remark. Keep in mind that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

The Heaviside Function In early calculus, most of us have probably encountered this function at least once. Also known as the step function, the Heaviside function is represented in piecewise form:

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

This function is particularly very useful for putting together or separating other functions. You will most commonly see it expressed as $u(t-a)$, where a is some constant. All it does is shift the function a units to the right. This function is 0 when $t < a$ and 1 when $t \geq a$.

For example, treat $f(t)$ as a signal. If you received the signal $\sin t$ at a time $t = \pi$, then the step function should be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Using the Heaviside function, we have $f(t) = u(t-\pi)\sin t$. The step function which evaluates to 1 on the interval $[1, 2)$ and zero outside this range can be written as

$$u(t-1) - u(t-2)$$

The Heaviside function is extremely useful to define piecewise functions. Hence, it is useful to know its Laplace transform. Using the table in section 7.1, we already know

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{2}$$

Problem 7.2.2. *Consider the hypothetical scenario where a forcing function is not periodic. That is, if we had a mass-spring system with differential equation*

$$x''(t) + x(t) = f(t)$$

with initial conditions $x(0) = 0$ and $x'(0) = 0$, where $f(t) = 1$ for $t \in [1, 5)$ and zero otherwise. What is the solution $x(t)$?

Solution. We could imagine a mass-spring system, where a rocket is fired for 4 seconds starting at $t = 1$. We can write the function as

$$f(t) = u(t-1) - u(t-5)$$

We aren't simplifying this for a reason. Taking the Laplace transform of both sides of the equation, we obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}$$

Solving for $X(s)$, we get

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}$$

It is an exercise for you to verify that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

Applying the linearity of the Laplace transform we obtain

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-s}\mathcal{L}\{1-\cos t\}\} = (1-\cos(t-1))u(t-1)$$

Similarly, linearity gives

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-5s}\mathcal{L}\{1-\cos t\}\} = (1-\cos(t-5))u(t-5)$$

So the general solution to this differential equation is

$$\boxed{x(t) = (1-\cos(t-1))\ u(t-1) - (1-\cos(t-5))\ u(t-5)}$$

Transfer Functions Often, in engineering, we are interested in analyzing the steady-state behavior of a linear system. Suppose we have an equation of type

$$Lx = f(t)$$

where L is a differential operator describing a constant-coefficient linear differential equation. $f(t)$ can be treated as a system input and $x(t)$ as the system output. We would like to have an convenient way to study the behavior of the system for different inputs.

To keep things simple, assume all initial conditions are zero. Taking the Laplace transform of both sides of the equation, we have

$$A(x)X(s) = F(s)$$

Rearranging the equation to solve for $\frac{X(s)}{F(s)}$, we obtain the **transfer function**

$$H(s) = \frac{1}{A(s)} = \frac{X(s)}{F(s)}$$

Observe that $X(s) = H(s)F(s)$. We now have an algebraic relationship between the output of the system with its input. We can now easily study the steady state behavior of the system given different inputs by simply multiplying by the transfer function.

Problem 7.2.3. *Assuming initial conditions of zero, use the transfer function for the differential equation $x''(t) + \omega_0^2 x(t) = f(t)$ to find the general solution.*

Solution. The first step is to take the Laplace transform of both sides of the equation.

$$s^2X(s) + \omega_0^2X(s) = F(s)$$

Solving for the transfer function, we have

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2+\omega_0^2}$$

How is this helpful to us? Suppose we have a constant input $f(t) = 1$. Hence, $F(s) = \mathcal{L}\{t\} = \frac{1}{s}$ and

$$X(s) = H(s)F(s) = \frac{1}{s^2+\omega_0^2}\frac{1}{s}$$

Finally, we take the inverse Laplace transform of $X(s)$ to obtain the general solution as

$$\boxed{x(t) = \frac{1-\cos(\omega_0 t)}{\omega_0^2}}$$

Laplace Transforms of Integrals Laplace transforms are special in the aspect that they can deal with integral equations, not only differential. These equations are those involving integrals rather than derivatives of functions. The basic property of such Laplace transforms can be easily derived using integration by parts, and it presents as

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

And since we have to take the inverse transform in so many scenarios, we can rewrite the equation as

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

Problem 7.2.4. *Consider the integral equation $t^2 = \int_0^t e^\tau x(\tau)\,d\tau$. Find the general solution $x(t)$.*

Solution. Taking the Laplace transform of the equation, we obtain

$$\frac{2}{s^3} = \frac{1}{s}\mathcal{L}\{e^t x(t)\} = \frac{1}{s}X(s-1)$$

We know that $X(s) = \mathcal{L}\{x(t)\}$. Therefore

$$X(s-1) = \frac{2}{s^3}$$

Applying the definition of shifting property, we can isolate for $X(s)$ as

$$X(s) = \frac{2}{(s+1)^2}$$

Applying the shifting property again, we get the general solution as

$$\boxed{x(t) = 2te^{-t}}$$

7.3 Convolution

What is Convolution? In section 7.1, we saw that the Laplace transform of a product is not necessarily equivalent to the product of the transforms. However, we're not doomed. We can work around this constraint by using a different type of "product." Take two continuous functions on the time interval, $f(t)$ and $g(t)$, defined for $t \geq 0$, so the **convolution** of $f(t)$ and $g(t)$ as

$$(f \star g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Interestingly enough, the convolution of two functions of t is also a function of t.

Problem 7.3.1. *Let $f(t) = e^t$ and $g(t) = t$, $t \geq 0$. Find the convolution of the two functions.*

Solution. We have

$$(f \star g)(t) = \int_0^t e^\tau (t-\tau)\, d\tau = \boxed{e^t - t - 1}$$

Note that we used integration by parts to compute this integral.

Problem 7.3.2. *Let $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ for $t \geq 0$. Find the convolution of the two functions.*

Solution. We first set up the integral.

$$(f \star g)(t) = \int_0^t \sin(\omega t)\ \cos(\omega(t-\tau))\, d\tau$$

We need to apply the trigonometric identity

$$\cos(\theta)\sin(\psi) = \frac{1}{2}\left(\sin(\theta+\psi) - \sin(\theta-\psi)\right)$$

So the convolution of $f(t)$ and $g(t)$ is

$$\begin{aligned}(f * g)(t) &= \int_0^t \frac{1}{2}(\sin(\omega t) - \sin(\omega t - 2\omega\tau))\, d\tau \\ &= \left[\frac{1}{2}\tau\sin(\omega t) + \frac{1}{4\omega}\cos(2\omega\tau - \omega t)\right]_{\tau=0}^t \\ &= \boxed{\frac{1}{2}t\sin(\omega t)}\end{aligned}$$

As expected, this formula holds only for $t \geq 0$. For values t outside this range, we assume that the functions $f(t)$ and $g(t)$ are zero.

The reason why we treat convolution as a type of "product" is because they have many properties that allow it to function similarly. For example, if $c \in \mathbb{R}$ is a constant and f, g, and h are functions, we have

$$\begin{aligned} f \star g &= g \star f \\ (cf) \star g &= f \star (cg) = c(f \star g) \\ (f \star g) \star h &= f \star (g \star h)\end{aligned}$$

The most interesting property arises when $f(t)$ and $g(t)$ are functions of exponential order. The convolution is then

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

We can observe that the Laplace transform of a convolution is equal to the product of the individual transforms. Usually, this result is used in the reverse. Let's look at an example.

Problem 7.3.3. *Using the definition of convolution, find the inverse Laplace transform of the function $F(s) = \frac{1}{(s+1)s^2}$.*

Solution. The first step is to express the function as a product of two simpler functions. We have

$$F(s) = \frac{1}{(s+1)s^2} = \frac{1}{s+1} \cdot \frac{1}{s^2}$$

From the Laplace transform table, we know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

We now apply the convolution property. If

$$F(s) = F_1(s)F_2(s)$$

for some continuous functions F_1 and F_2, then we have

$$\mathcal{L}^{-1}\{F(s)\} = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$$

where $f_1(t) = e^{-t}$ and $f_2(t) = t$. Therefore, the integral becomes

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)s^2}\right\}\} = \int_0^t \tau e^{-(t-\tau)}\,d\tau$$

Since we are integrating with respect to τ, we can take out e^{-t} as a constant, giving $e^{-t}\int_0^t \tau e^{\tau}\,d\tau$. We need to use integration by parts to compute this. Let $u = \tau$ and $dv = e^{\tau}\,d\tau$. We have $du = d\tau$ and $v = e^{\tau}$. Therefore, we have

$$\begin{aligned}\int \tau e^{\tau}\,d\tau &= \tau e^{\tau} - \int e^{\tau}\,d\tau \\ &= \tau e^{\tau} - e^{\tau} \\ &= e^{\tau}(\tau - 1)\end{aligned}$$

If we run the antiderivative from 0 to t, we have

$$\begin{aligned}\int_0^t \tau e^{\tau}\,d\tau &= [e^{\tau}(\tau-1)]_0^t \\ &= e^t(t-1) - e^0(0-1) \\ &= e^t(t-1) + 1\end{aligned}$$

The final step is to multiply by e^{-t}, because we temporarily pulled it out of the integrand. So the final answer is

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)s^2}\right\} = e^{-t}[e^t(t-1)+1] = \boxed{t - 1 + e^{-t}}$$

Finding Solutions to Ordinary Differential Equations The combination of the convolution and the Laplace transform is particularly most powerful when applied to solving ODEs. Especially in the physics field, we can provide the solution to the forced oscillation problem for any forcing function as a definite integral.

Problem 7.3.4. *Solve the differential equation $x'' + \omega_0^2 x = f(t)$, with initial conditions $x(0) = 0$ and $x'(0) = 0$, where $f(t)$ is an arbitrary function.*

Solution. The first step is to apply the Laplace transform to $x(t)$. We denote the transform of $x(t)$ as $X(s)$ and the transform of $f(t)$ as $F(s)$. We have

$$s^2X(s) + \omega_0^2X(s) = F(s)$$

We can isolate for $X(s)$ by factoring the left side of the equation.

$$X(s) = F(s)\frac{1}{s^2 + \omega_0^2}$$

We know that the inverse Laplace transform of $\frac{1}{s^2+\omega_0^2}$ is

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega_0^2}\right\} = \frac{\sin(\omega_0 t)}{\omega_0}$$

Therefore, by convolution, we have

$$x(t) = \int_0^t f(\tau)\frac{\sin(\omega_0(t-\tau))}{\omega_0}\,d\tau$$

We can reverse the order of the convolution to obtain

$$x(t) = \int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}f(t-\tau)\,d\tau$$

Suppose $f(t) = \cos(\omega_0 t)$. Then the integral becomes

$$x(t) = \int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}\cos(\omega_0(t-\tau))\,d\tau = \frac{1}{\omega_0}\int_0^t \sin(\omega_0\tau)\cos(\omega_0(t-\tau))\,d\tau$$

We already know from a previous example the convolution of the sine and cosine functions. Hence, the general solution is

$$x(t) = \left(\frac{1}{\omega_0}\right)\left(\frac{1}{2}t\sin(\omega_0 t)\right) = \boxed{\frac{1}{2\omega_0}\sin(\omega_0 t)}$$

Remark. Observe the t in front of the sine. The solution grows without bound as t gets sufficiently large. This mechanism is called *resonance*.

We can solve pretty much any constant-coefficient ODE with an arbitrary forcing function $f(t)$ applying the convolution process. Instead of a closed form solution, a definite integral serves as sufficient in most practical purposes, as it is generally not very hard to numerically evaluate one.

Volterra Integral Equation The last topic of this section is a special integrating equation named after the Italian mathematician Vito Volterra (1860-1940). The equation presents as

$$x(t) = f(t) + \int_0^t g(t-\tau)x(\tau)\,d\tau$$

where $f(t)$ and $g(t)$ are unknown functions and we wish to find the solution for $x(t)$. If we apply the Laplace transform to both sides of the equation, we are left with

$$X(s) = F(s) + G(s)X(s)$$

where $X(s)$, $F(s)$, and $G(s)$ are the Laplace transforms of the functions $x(t)$, $f(t)$, and $g(t)$, respectively. Isolating for $X(s)$, we get

$$X(s) = \frac{F(s)}{1 - G(s)}$$

Now we should understand that in order to obtain the solution to $x(t)$, we need to apply the inverse Laplace transform to $X(s)$.

Problem 7.3.5. *Solve the equation*

$$x(t) = e^{-t} + \int_0^t \sinh(t-\tau)x(\tau)\,d\tau$$

Solution. Applying the Laplace transform to the equation, we get

$$X(s) = \frac{1}{s+1} + \frac{1}{s^2-1}X(s)$$

Using some algebraic manipulation, we write the right hand side as

$$X(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}$$

If we reference the Laplace transform table provided in section 7.1, we find that the general solution for $x(t)$ is

$$\boxed{x(t) = \cosh(\sqrt{2}t) - \frac{1}{\sqrt{2}}\sinh(\sqrt{2}t)}$$

7.4 Dirac Delta and Impulse Response

The reason we first discussed Heaviside, or step functions, was because they can be thought of as switches changing the forcing function $f(t)$ at specific times. However, these functions are not very effective to forcing functions that generate large force magnitudes in very short time frames.

Introduction to Dirac Delta For example, consider a hammer striking a nail or a short circuit in an electrical system. These are significant forces, but their time intervals are extremely short. This is where we introduce the **Dirac Delta function** to deal with such forcing functions.

The Dirac Delta function consists of three major properties:

1. $\delta(t-a) = 0$, for $t \neq a$
2. $\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)\,dt = 1$, for $\varepsilon > 0$
3. $\int_{a-\varepsilon}^{a+\varepsilon} f(t)\delta(t-a)\,dt = f(a)$, for $\varepsilon > 0$

From this series of properties, it follows that the Dirac Delta function is "infinite" in a sense when $t = a$. So, the Dirac Delta function is a function that is zero everywhere except for one specific point.

Dirac Delta: Function or Not? The Dirac Delta is a rather odd structure. It is zero everywhere except one point and yet the integral of any interval containing that one point has a value of 1. It is not really a function, in the way we think. Rather, it would make more sense to call it a *distribution*.

Despite its erratic behavior, it is very useful for modeling sudden shocks or sufficiently large forces to a physical system.

Solving Initial Value Problems (IVPs) We can solve initial value problems for complex differential equations by transforming the Dirac Delta function. Using the third property, we get

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$$

provided that a is a positive value. Let's try our hands at a problem.

Problem 7.4.1. *Solve the following IVP with initial conditions given below:*

$$y'' + 2y' - 15y = 6\delta(t-9) \qquad y(0) = -5 \qquad y'(0) = 7$$

Solution. Take the Laplace transform of both sides of the equation to get

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 15Y(s) = 6e^{-9s}$$

Factoring the equation, we obtain

$$(s^2 + 2s - 15)Y(s) + 5s + 3 = 6e^{-9s}$$

so solving for $Y(s)$ yields

$$\begin{aligned} Y(s) &= \frac{6e^{-9s}}{(s+5)(s-3)} - \frac{5s+3}{(s+5)(s-3)} \\ &= 6e^{-9s}F(s) - G(s) \end{aligned}$$

It is left as a challenge to you to verify the partial fraction decomposition and the inverse Laplace transforms as

$$\begin{aligned} F(s) &= \frac{1}{(s+5)(s-3)} = \frac{1/8}{s-3} - \frac{1/8}{s+5} \\ f(t) &= \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t} \\ G(s) &= \frac{5s+3}{(s+5)(s-3)} = \frac{9/4}{s-3} + \frac{11/4}{s+5} \\ g(t) &= \frac{9}{4}e^{3t} + \frac{11}{4}e^{-5t} \end{aligned}$$

So the solution is then $\boxed{y(t) = 6u_9(t)f(t-9) - g(t)}$, where $f(t)$ and $g(t)$ were defined above.

Here's a slightly more complex problem. But don't worry, because the approach is the same.

Problem 7.4.2. *Solve the following IVP with initial conditions given below:*

$$2y'' + 10y = 3u_{12}(t) - 5\delta(t-4) \qquad y(0) = -1 \qquad y'(0) = -2$$

Solution. Of course, we start by taking the Laplace transform of the differential equation and applying the correct initial conditions.

$$2(s^2Y(s) - sy(0) - y'(0)) + 10Y(s) = \frac{3e^{-12s}}{s} - 5e^{-4s}$$

Simplifying the problem, we have

$$(2s^2 + 10)Y(s) + 2s + 4 = \frac{3e^{-12s}}{s} - 5e^{-4s}$$

Solving for $Y(s)$, we get

$$\begin{aligned} Y(s) &= \frac{3e^{-12s}}{s(2s^2+10)} - \frac{5e^{-4s}}{2s^2+10} - \frac{2s+4}{2s^2+10} \\ &= 3e^{-12s}F(s) - 5e^{-4s}G(s) - H(s) \end{aligned}$$

For the first fraction, we will need to apply partial fraction decomposition. The other two are left as challenges to you to apply u-substitution and a little algebraic manipulation. We have

$$F(s) = \frac{1}{s(2s^2+10)} = \frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s}{s^2+5}$$

$$f(t) = \frac{1}{10} - \frac{1}{10}\cos(\sqrt{5}t)$$

$$g(t) = \frac{1}{2\sqrt{5}}\sin(\sqrt{5}t)$$

$$h(t) = \cos(\sqrt{5}t) + \frac{2}{\sqrt{5}}\sin(\sqrt{5}t)$$

So the general solution is $\boxed{y(t) = 3u_{12}(t)f(t-12) - 5u_4(t)g(t-4) - h(t)}$, where $f(t)$, $g(t)$, and $h(t)$ were defined above.

Remark. Aside from the new function, these problems operate the same way as the ones we've worked through so far. It's also important to notice that the exponential term was brought into the transform by the Dirac Delta function, but once it's part of the transform, its origin no longer matters. When we computed the inverse transform, it reappeared as a Heaviside function.

Let's end this section with a reflection on the Heaviside function. We can relate it to the Dirac Delta. Start with the integral

$$\int_{-\infty}^{t} \delta(u-a)\,du = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

This is precisely the basic definition of the Heaviside function. So

$$\int_{-\infty}^{t} \delta(u-a)\,du = u_a(t)$$

If we apply the Fundamental Theorem of Calculus, we obtain

$$u_a'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-a)\,du\right) = \delta(t-a)$$

So we can observe that the Heaviside function is just the derivative of the Dirac Delta function!

7.5 Transforms of Partial Differential Equations

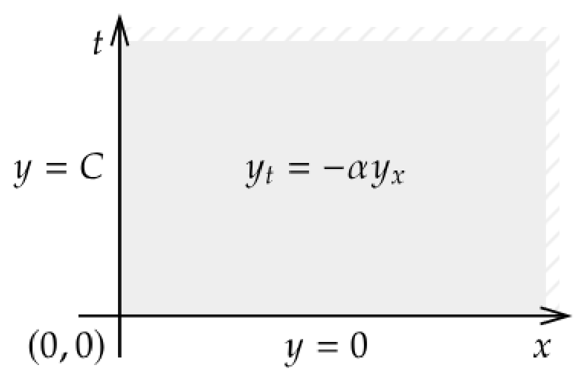
A *partial differential equation* (PDE) involves a dependent variable in terms of more than one independent variable. Call such variables x and t.

Key Idea We can use the Laplace transform on one of the variables at a time, and as expected, derivatives in terms of that variable become multiplications by the transformed variable s. The PDE ultimately becomes an ODE, which we can solve using known methods. Once finished, we take the inverse transform to find the general solution to the original problem.

Let's consider the first-order partial differential equation

$$y_t = -\alpha y_x$$

defined for $x > 0$ and $t > 0$. We also have side conditions $y(0,t) = C$ and $y(x,0) = 0$. In heat transfer, this equation is referred to as the *convection equation*. On a half line, the setup looks something like this:



Consider a physical model in which a river of thick, non-diffusive goo flows downstream, so we ignore any diffusion effects. Let y represent the concentration of a toxic substance in the river. The spatial variable x denotes position along the river, with $x = 0$ marking the location of a factory that continuously releases the toxin into the water. The factory maintains a constant concentration C at $x = 0$. Our goal is to study the behavior of the concentration for points downstream, i.e., for $x > 0$. Let t represent time, and suppose the factory begins operating at $t = 0$. At that initial time, the river contains no toxin and consists entirely of pure goo.

Now consider a function of two variables $y(x,t)$. Let's hold x constant and then transform the t variable. For convenience, we treat the transformed variable s as a parameter because there are no such derivatives in s. That is, we place $Y(x)$ for the transformed function, treating it as a function of x, leaving s as a simple parameter.

$$Y(x) = \mathcal{L}\{y_x(x,t)\} = \int_0^\infty y_x(x,t)e^{-st}\,ds$$

The transform of a derivative with respect to x is just differentiating the transformed function:

$$\mathcal{L}\{y_x(x,t)\} = \int_0^\infty y_x(x,t)e^{-st}\,ds = \frac{d}{dx}\left[\int_0^\infty y(x,t)e^{-st}\,ds\right] = Y'(x)$$

In order to transform the derivative in t, we use the rules from section 7.2 to get

$$\mathcal{L}\{y_t(x,t)\} = sY(x) - y(x,0)$$

Specifically, we have $y(x,0) = 0$ and so $\mathcal{L}\{y_t(x,t)\} = sY(x)$. Transforming this equation yields us

$$sY(x) = -\alpha Y'(x)$$

This is an ODE. We are missing an initial condition, because it is located on the other side condition of the PDE, which depends entirely on x. Everything has been transformed, so we need to transform the initial condition as well.

$$Y(0) = \mathcal{L}\{y(0,t)\} = \mathcal{L}\{C\} = \frac{C}{s}$$

Now we can solve the ODE problem, $sY(x) = -aY'(x)$, with $Y(0) = \frac{C}{s}$, to find that

$$Y(x) = \frac{C}{s}e^{-\frac{s}{a}x}$$

We are not done, however, because we wish to determine $y(x,t)$. We have to transform the s variable back to t. Let's now define the Heaviside function.

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{otherwise} \end{cases}$$

We know

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{s}$$

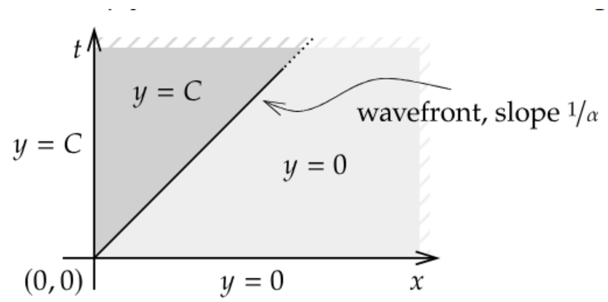
so the general solution becomes

$$\begin{aligned} y(x,t) &= \mathcal{L}^{-1}\left\{\frac{C}{s}e^{-\frac{s}{\alpha}x}\right\} \\ &= \boxed{Cu(t - x/\alpha)} \end{aligned}$$

We could also express the solution in piecewise form. Let

$$y(x,t) = \begin{cases} 0 & \text{if } t < x/\alpha, \\ C & \text{otherwise} \end{cases}$$

Below shows the diagram of this solution in the first quadrant. The line of slope $1/\alpha$ indicates the wavefront of the toxic substance in the image as it is leaving the factory. The initial condition simply shifts the initial condition to the right at a speed (rate) of α.



Here's a fact which is rather interesting. y is neither continuous nor differentiable. So how could we possibly plug something not differentiable into the equation. The trick is recognizing the fact that the derivative of the Heaviside function is the Dirac Delta function, so

$$y_t(x,t) = \frac{\partial}{\partial t}\left[Cu(t - x/\alpha)\right] = Cu'(t - x/\alpha) = C\delta(t - x/\alpha)$$

and similarly, we have

$$y_x(x,t) = \frac{\partial}{\partial x}\left[Cu(t - x/\alpha)\right] = -\frac{C}{\alpha}u'(t - x/\alpha) = -\frac{C}{\alpha}\delta(t - x/\alpha)$$

so via inspection, it is clear that $y_t = -\alpha y_x$. □

The Laplace transform is very powerful to solve constant-coefficient differential equations. It also easily handles non-homogeneous differential equations. Let's try a more difficult example.

Problem 7.5.1. *Consider the system of differential equations shown below.*

$$y_t + y_x + y = 0, \quad x > 0, t > 0$$
$$y(0,t) = \sin(t), \quad y(x,0) = 0$$

Find the general solution $y(x,t)$.

Solution. We transform the variable t and then write $Y(x)$ for the transformed function. Since $y(x,0) = 0$, we get

$$sY(x) + Y'(x) + Y(x) = 0, \quad Y(0) = \frac{1}{s^2+1}$$

and the solution to the transformed equation is

$$Y(x) = \frac{1}{s^2+1}e^{-(s+1)x} = \frac{1}{s^2+1}e^{-xs}e^{-x}$$

Using the second shifting property from section 7.1 and the principle of linearity, we find the general solution as

$$\boxed{y(x,t) = e^{-x}\sin(t-x)u(t-x)}$$

Problem 7.5.2. *Find the general solution to the following system of equations shown below.*

$$y_t = y_{xx}, \quad 0 < x < \infty,\ t > 0$$
$$y_x(0,t) = f(t)$$
$$y(x,0) = 0$$

Solution. Let's impose that y is bounded for each fixed time t. We transform the equation in t to obtain

$$sY(x) = Y''(x)$$

and the general solution to this ODE is

$$Y(x) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$$

Clearly, $A = 0$ because otherwise Y does not decay to zero as $s \to \infty$. Now consider the boundary condition. Transform $Y'(0) = F(s)$ and so $-\sqrt{s}B = F(s)$. So in other words,

$$Y(x) = -F(s)\frac{1}{\sqrt{s}}e^{-\sqrt{s}x}$$

If we look up the Laplace transform table, we can take the inverse transform to get

$$\mathcal{L}^{-1}\{e^{-\sqrt{s}x}\} = \frac{x}{\sqrt{4\pi t^3}}e^{\frac{-x^2}{4t}}$$

Thus, the general solution to the differential equation is

$$\begin{aligned} y(x,t) &= \mathcal{L}^{-1}\{F(s)e^{-\sqrt{s}x}\} \\ &= \boxed{\int_0^t f(\tau)\frac{1}{\sqrt{\pi(t-\tau)}}e^{\frac{-x^2}{4(t-\tau)}}\,d\tau} \end{aligned}$$

Remark. The Laplace transform method is useful for handling differential equations that cannot be solved using separation of variables. It can easily deal with nonhomogeneous terms, but it is mainly effective for equations with constant coefficients.

Thank you so much for reading this book! I am honored to have contributed to your academic journey in some way!

Thanks,

Aditya Baisakh

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